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Research Article

A Note on Closed-Form Representation of Fibonacci Numbers Using Fibonacci Trees

Indhumathi Raman

School of Information Technology and Engineering, VIT University, Vellore 632014, India

Correspondence should be addressed to Indhumathi Raman; indhumathi.raman@gmail.com

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We give a new representation of the Fibonacci numbers. This is achieved using Fibonacci trees. With the help of this representation, the *n*th Fibonacci number can be calculated without having any knowledge about the previous Fibonacci numbers.

1. Introduction

A Fibonacci tree is a rooted binary tree in which for every nonleaf vertex v, the heights of the subtrees, rooted at the left and right child of v, differ by exactly one. A formal recursive definition of the Fibonacci tree (denoted by $\mathbb{F}_h$ if its height is h) is given below.

Definition 1. $\mathbb{F}_0 := K_1, \mathbb{F}_1 := K_2$. For $h \geq 2$, $\mathbb{F}_h$ is obtained by taking a copy of $\mathbb{F}_{h-1}$, a copy of $\mathbb{F}_{h-2}$, a new vertex R, and joining R to the roots of $\mathbb{F}_{h-1}$ and $\mathbb{F}_{h-2}$.

Figure 1 shows this construction and a few small Fibonacci trees.

The above recursive definition implies that the number of vertices in $\mathbb{F}_h$ is $|V(\mathbb{F}_h)| = |V(\mathbb{F}_{h-1})| + |V(\mathbb{F}_{h-2})| + 1$. On solving this recurrence relation, we get $|V(\mathbb{F}_h)| = f(h+2) - 1$, where $f(i)$ is the ith number in the Fibonacci sequence, $f(0) = 1$, $f(1) = 1$, $f(n) = f(n-1) + f(n-2)$; this justifies the terminology Fibonacci tree.

The Fibonacci tree is the one with the minimum number of vertices among the class of AVL trees (see [1]). Several properties of Fibonacci trees have been investigated: for example, Fibonacci numbers of Fibonacci trees have been studied in [2], optimality of Fibonacci numbers is discussed in [3], asymptotic properties of Balaban's index for Fibonacci trees have been explored in [4], and Zeckendorf representation of integers is given in [5]. In this short paper, we represent the number of vertices of $\mathbb{F}_h$ in *closed form* (A closed form is one which gives the value of a sequence at index n in terms of only one parameter, n itself.) by observing the number of vertices at each level of $\mathbb{F}_h$. Such a calculation helps us to give a closed-form representation of nth Fibonacci number for every $n \geq 2$.

2. Closed-Form Representation of Fibonacci Numbers

There are several closed-form representations of the Fibonacci numbers. We state a few below.

(i) Consider

$$f(n) = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}. \quad (1)$$

It was also derived by Binet (see [6]) in 1843, although the result was known to Euler, Daniel Bernoulli, and de Moivre more than a century earlier.

(ii) Consider

$$B(x) = \sum_{k=0}^{\infty} b_k x^k. \quad (2)$$

In the above generating function for the Fibonacci numbers the value of b_k gives the kth Fibonacci number. However, expanding the generating function involves tedious calculations.

(iii) Consider

$$f_n = \text{round}\left(\frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n\right). \tag{3}$$

It was also derived by Binet (see [6]) where the function round() rounds the simplified expression up or down to an integer.

In this section, we give a simpler closed-form combinatorial representation of Fibonacci numbers. To do so, we first give a closed-form representation of the number of vertices $|V(\mathbb{F}_h)|$ of $\mathbb{F}_h$ (the Fibonacci tree of height h). The following lemma gives the number of vertices in a particular level of $\mathbb{F}_h$ and thereafter we sum the number of vertices over the levels to get $|V(\mathbb{F}_h)|$.

Lemma 2. *Let $\mathbb{F}_h$ be a Fibonacci tree of height h and let k be an integer such that $0 \le k \le h$. The number of vertices $N(h,k)$ at level k of $\mathbb{F}_h$ is given by*

$$N(h,k) = \sum_{i=0}^{h-k} \binom{k}{h-k-i}. \tag{4}$$

Proof. We prove the lemma by induction on k. For $k = 0$ we have $N(h,0) = \sum_{i=0}^{h} \binom{0}{h-i}$. Using the convention $\binom{n}{r} = 0$ if $n < r$, we have $N(h,0) = \binom{0}{0} = 1$. This is true since the root of $\mathbb{F}_h$ is the only vertex at level 0. Further proceeding, from the recursive definition of $\mathbb{F}_h$, we have

$$\begin{aligned}
&N(h,k) \\
&= N(h-1,k-1) + N(h-2,k-1) \\
&= \sum_{i=0}^{h-k} \binom{k-1}{h-k-i} + \sum_{j=0}^{h-k-1} \binom{k-1}{h-k-j-1} \\
&= \sum_{i=0}^{h-k} \binom{k-1}{h-k-i} + \sum_{j=0}^{h-k} \binom{k-1}{h-k-j-1} \\
&\quad - \binom{k-1}{-1} \\
&= \sum_{i=0}^{h-k} \left(\binom{k-1}{h-k-i} + \binom{k-1}{h-k-i-1}\right) \quad \text{since} \binom{n}{r} = 0 \\
&\qquad\qquad\qquad\qquad \text{if } r < 0 \\
&= \sum_{i=0}^{h-k} \binom{k}{h-k-i}.
\end{aligned} \tag{5}$$

In Step 3 of the above equation, we add and subtract $\binom{k-1}{h-k-j-1}$ for $j = h-k$. This proves the lemma. ☐

The number of vertices in any tree is the sum of the vertices at its levels. In particular, $|V(\mathbb{F}_h)| = \sum_{k=0}^{h} N(h,k)$. Hence we have the following lemma.

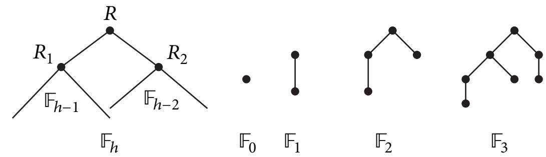

Figure 1: Recursive construction and examples of Fibonacci Trees.

Lemma 3. *Let $\mathbb{F}_h$ be the Fibonacci tree of height h; then the number of vertices $|V(\mathbb{F}_h)|$ of $\mathbb{F}_h$ is $\sum_{k=0}^{h}\sum_{i=0}^{h-k}\binom{k}{h-k-i}$.*

The above theorem helps us to derive a closed-form representation of the Fibonacci numbers. This representation is in contrast to the recurrence relation form, which has certain previous values of the sequence as parameters. We know that $|V(\mathbb{F}_h)| = f(h+2) - 1$. Equivalently $f(n) = 1 + |V(\mathbb{F}_{n-2})|$.

Theorem 4. *Let $f(n)$ be the nth number in the Fibonacci sequence starting with $f(0) = 1$ and $f(1) = 1$. Then for $n \ge 2$,*

$$f(n) = 1 + \sum_{k=0}^{n-2}\sum_{i=0}^{n-k-2} \binom{k}{n-k-i-2}. \tag{6}$$

Proof. Since $f(n) = |V(\mathbb{F}_{\{n-2\}})| + 1$, the proof is an immediate consequence of Lemma 3. ☐

As an example for Theorem 4, we calculate $f(4)$ and $f(5)$:

$$\begin{aligned}
f(4) &= 1 + \sum_{k=0}^{2}\sum_{i=0}^{2-k}\binom{k}{2-k-i} \\
&= 1 + \sum_{i=0}^{2}\binom{0}{2-i} + \sum_{i=0}^{1}\binom{1}{1-i} + \sum_{i=0}^{0}\binom{2}{0-i} \\
&= 1 + \binom{0}{0} + \binom{1}{1} + \binom{1}{0} + \binom{2}{0} \\
&= 5, \\
f(5) &= 1 + \sum_{k=0}^{3}\sum_{i=0}^{3-k}\binom{k}{3-k-i} \\
&= 1 + \sum_{i=0}^{3}\binom{0}{3-i} + \sum_{i=0}^{2}\binom{1}{2-i} \\
&\quad + \sum_{i=0}^{1}\binom{2}{1-i} + \sum_{i=0}^{0}\binom{3}{0-i} \\
&= 1 + \binom{0}{0} + \binom{1}{1} + \binom{1}{0} + \binom{2}{1} + \binom{2}{0} + \binom{3}{0} \\
&= 8.
\end{aligned} \tag{7}$$

3. Conclusion

In this paper, we give a closed-form representation of Fibonacci numbers using Fibonacci trees. A similar approach can be attempted for finding a closed-form representation for Lucas and Bernoulli numbers.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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