Indian Journal of Science and Technology, *Vol 8(20), DOI:10.17485/ijst/2015/v8i20/79986 August 2015*
*ISSN (Print) : 0974-6846*
*ISSN (Online) : 0974-5645*

# Field Energy Approach for Homogenization of Metamaterials

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## Abstract

**Objectives:** In recent years metamaterials have been widely accepted to explore its new and unique electromagnetic properties. Metamaterials differ from conventional materials in terms of its structure i.e. they have inclusions in place of atoms and molecules. It is because of this uncommon structure and geometry that metamaterials show these electromagnetic properties. **Methods:** To know about these responses in detail there is a nee d to find parameters like permittivity and permeability for these materials. To simplify this complicated process of calculation of parameters, homogenization techniques are used. **Findings:** So, this paper deals with the study of homogenization technique called Field Energy Method and retrieval of parameters from it. It also focuses on comparing Field Energy Method with Average Method and proving that earlier is more efficient than the later.

**Keywords:** Average Method, Electromagnetic Waves, Field Energy Method, Homogenization, Metamaterial

## 1. Introduction

Some special materials, called metamaterials can be defined as the synthetically made structures used to guide and influence light. They are unfamiliar complex materials which show properties superior to those available in already existing conventional materials. This makes metamaterials unique and different from normally occurring materials defined above. Unlike conventional materials, metamaterials show unusual properties because firstly, they are made of inclusions instead of atoms and molecules and size of these inclusions are lesser compared to the desired electromagnetic wavelength. Secondly, light[3] which is defined as an electromagnetic radiation and a way by which information is send to and from internal arrangement of materials, acts as one-handed on interaction with the constituents (atoms and molecules) of conventional materials because only one of its component i.e. electric field investigates the internal structure of the material efficiently and magnetic field remains unused, whereas light depicts two-handed nature for logically designed metamaterials i.e. both the field components are coupled to meta-atoms strongly and interact with them effectively.

One unique property which arises from this two-handed interaction is negative refractive index[4]. Super lens is one of the best examples which have been developed by the metamaterials displaying negative refractive index. Super lens has the potential of imaging rare and superior structures that are smaller than the wavelength of light. There are many other applications[5] of metamaterials, metacoatings is one of them. It is responsible for the invisibility of objects i.e. it makes object invisible by guiding light rays around the object. If it works, it will be more effective than any of the currently existing "invisibility cloaking" devices, based on optical sensors and computers.

Since, the responses of the inclusions, as well as their interactions with electromagnetic waves, can often be incorporated into continuous, effective material parameters. So, there is a need to work on obtaining parameters like permittivity and permeability of these materials to know about their properties more accurately. Permittivity can be defined as the measurement of resistance while forming an electric field in a material medium. With increase in permittivity value, amount of electric field or electric flux decreases in any material. Similarly, permeability can be defined as the measurement

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of the resistance while forming magnetic field in a material medium.

Even though permittivity (€) and permeability (μ) decide the properties of materials, but, determining their values is not easy as fields are not uniform and they vary at every instant of time. Such fields are called local fields. So, there is a need to uniform fields so that € and μ values can easily be determined. This method of converting local fields to macroscopic fields by making them uniform is called homogenization[8].

Homogenization can be applied, when the light or desired EM (Electromagnetic) waves have wavelength more than the size of the constituents of metamaterials. There are various homogenization techniques like Average Method. But, this paper deals with a new homogenization technique, known as Field Energy Method, which overcomes one major drawback of Average method. In Average Method, energy of the whole system is not same i.e. energy of the homogenized field is less than the energy of local field and for any system to function it is necessary that energy of whole system should be maintained. This need is fulfilled by Field Energy Method.

## 2. Methodology

### 2.1 Differences in Field Energy Method and Average Method

This paper deals with the study of Field Energy Method and how it is better than the Average Method. In Field Energy Method, energy of local fields is calculated which is then averaged. From this averaged field energy, effective field is derived to obtain the expressions for parameters € and μ.

Figure 1 depicts the methodology followed in Field Energy Method whereas in Average Method[1], local fields are directly averaged without calculating their energies to get the parameters. Figure 2 shows the process followed in case of Average method.

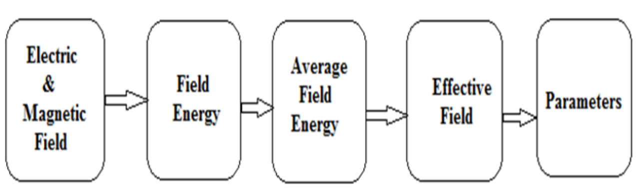


**Figure 1.** Field energy method.


**Figure 2.** Average method.

#### *2.1.1 Field Energy Approach*

To develop the homogenization theory for metamaterials by Field Energy Method, expressions for local fields, linked with propagating mode are found by solving the Maxwell's equations given below:

$\nabla \times E = jwB$ (1)

$\nabla \times H = jwD$ (2)

Where E is Electric Field, B stands for Magnetizing Field, D for Displacement field and H for Magnetic Field. All these fields form local fields. For this, we consider the unshaded cube with each side of 2d length as shown in Figure 3.

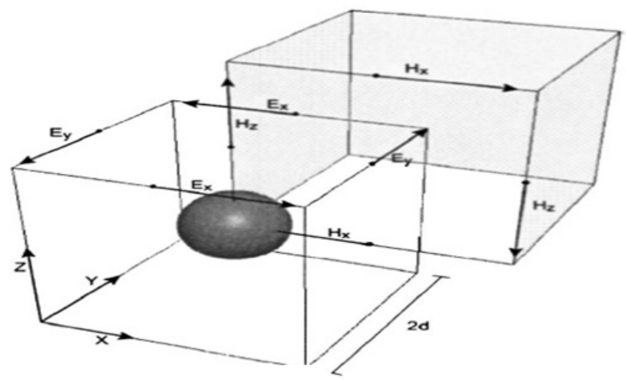


**Figure 3.** A periodic unit cell1 of length 2d with homogenized electric field on edges of first and homogenized magnetic field on edges of other cubic cell.

Apart from this, sphere located within the cube characterizes contents of cubic unit cell which is a periodic structure and repeats itself infinitely over whole material.

In region, inside cube and outside the inclusion relationship among $E_x$, $E_y$ and $B_z$ fields is given by Maxwell's Equation 1 as:

$$\left\{\frac{\partial(E_z)}{\partial y} - \frac{\partial(E_y)}{\partial z}\right\} = jwB_x \quad (3)$$

$$\left\{\frac{\partial(E_x)}{\partial z} - \frac{\partial(E_z)}{\partial x}\right\} = jwB_y \quad (4)$$

$$\int_{-d}^{d} E_z(d,d,z)dz - \int_{-d}^{d} E_z(d,-d,z)dz - \int_{-d}^{d} E_y(d,y,d)dy + \int_{-d}^{d} E_y(d,y,-d)dy = jw\int_{-d}^{d} dy \int_{-d}^{d} dz.\, B_x(d,y,z) \quad (6)$$

$$\int_{-d}^{d} E_x(x,d,d)dx - \int_{-d}^{d} E_x(x,d,-d)dx - \int_{-d}^{d} E_z(d,d,z)dz + \int_{-d}^{d} E_z(-d,d,z)dz = jw\int_{-d}^{d} dx \int_{-d}^{d} dz.\, B_y(x,d,z) \quad (7)$$

$$\int_{-d}^{d} E_y(d,y,d)dy - \int_{-d}^{d} E_y(-d,y,d)dy - \int_{-d}^{d} E_x(x,d,d)dx + \int_{-d}^{d} E_x(x,-d,d)dx = jw\int_{-d}^{d} dx \int_{-d}^{d} dy.\, B_z(x,y,d) \quad (8)$$

$$\left\{\frac{\partial(E_y)}{\partial x} - \frac{\partial(E_x)}{\partial y}\right\} = jwB_z \quad (5)$$

We obtain Equations 6, 7 and 8 when Equations 3, 4 and 5 are integrated over positive surfaces of x, y and z, respectively such that, x = y = z = +d.

Homogenized field expressions[1] derived for Average Method by using general definitions of same are:

$E_{ef\,z}$ (d,d,0) $-E_{ef\,z}$ (d,-d,0) - $E_{ef\,y}$ (d,0,d) + $E_{ef\,y}$ (d,0,-d) = jw(2d) $B_{ef\,x}$ (d,0,0) (9)

$E_{ef\,x}$ (0,d,d) - $E_{ef\,x}$ (0,d,-d) - $E_{ef\,z}$ (d,d,0) $+E_{ef\,z}$ (-d,d,0)=jw(2d) $B_{ef\,y}$ (0,d,0) (10)

$E_{ef\,x}$ (0,d,d) - $E_{ef\,x}$ (0,-d,d) - $E_{ef\,y}$ (d,0,d) + $E_{ef\,y}$ (-d,0,d) =jw(2d) $B_{ef\,z}$ (0,0,d) (11)

Energy Approach presents itself with the help of following definitions:

$$E_{efx}(0,\pm d,d) = \sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_x(x,\pm d,d)\right|^2.\,dx} \quad (12)$$

$$E_{efy}(\pm d,0,d) = \sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_y(\pm d,y,d)\right|^2.\,dy} \quad (13)$$

$$B_{efz}(0,0,d) = \sqrt{\frac{1}{4d^2}\int_{-d}^{d}\int_{-d}^{d}\left|B_z(x,y,d)\right|^2 dx.\,dy} \quad (14)$$

$$\sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_z(d,+d,z)\right|^2 dz} - \sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_z(d,-d,z)\right|^2 dz} - \sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_y(d,y,+d)\right|^2 dy} + \sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_y(d,y,-d)\right|^2 dy} = jw(2d)\sqrt{\frac{1}{4d^2}\int_{-d}^{d}\int_{-d}^{d}\left|B_x(d,y,z)\right|^2 dy.\,dz} \quad (15)$$

$$\sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_x(x,+d,d)\right|^2 dx} - \sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_x(x,-d,d)\right|^2 dx} - \sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_y(+d,y,d)\right|^2 dy} + \sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_y(-d,y,d)\right|^2 dy} = jw(2d)\sqrt{\frac{1}{4d^2}\int_{-d}^{d}\int_{-d}^{d}\left|B_z(x,y,d)\right|^2 dx.\,dy} \quad (16)$$

$$\sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_x(x,d,+d)\right|^2 dx} - \sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_x(x,d,-d)\right|^2 dx} - \sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_z(+d,d,z)\right|^2 dz} + \sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_z(-d,d,z)\right|^2 dz} = jw(2d)\sqrt{\frac{1}{4d^2}\int_{-d}^{d}\int_{-d}^{d}\left|B_x(x,d,z)\right|^2 dx.\,dz} \quad (17)$$

$$\left\{\frac{\partial(H_Z)}{\partial y} - \frac{\partial(H_y)}{\partial z}\right\} = -jwD_x \quad (18)$$

$$\left\{\frac{\partial(H_X)}{\partial z} - \frac{\partial(H_z)}{\partial x}\right\} = -jwD_y \quad (19)$$

$$\left\{\frac{\partial(H_y)}{\partial x} - \frac{\partial(H_x)}{\partial y}\right\} = -jwD_z \quad (20)$$

$$\int_{0}^{2d} H_z(0,2d,z)\,dz - \int_{0}^{2d} H_z(0,0,z)\,dz - \int_{0}^{2d} H_y(0,y,2d)\,dy + \int_{0}^{2d} H_y(0,y,0)\,dy = jw\int_{0}^{2d} dy \int_{0}^{2d} dz.\,D_x(0,y,z) \quad (21)$$

$$\int_{0}^{2d} H_x(x,0,2d)\,dx - \int_{0}^{2d} H_x(x,0,0)\,dx - \int_{0}^{2d} H_z(2d,0,z)\,dz + \int_{0}^{2d} H_z(0,0,z)\,dz = jw\int_{0}^{2d} dx \int_{0}^{2d} dz.\,D_y(x,0,z) \quad (22)$$

$$\int_{0}^{2d} H_y(2d,y,0)\,dy - \int_{0}^{2d} H_y(0,y,0)\,dy - \int_{0}^{2d} H_x(x,2d,0)\,dx + \int_{0}^{2d} H_x(x,2d,0)\,dx = jw\int_{0}^{2d} dx \int_{0}^{2d} dy.\,D_z(x,y,0) \quad (23)$$

$$H_{efz}(0,2d,d) - H_{efz}(0,0,d) - H_{efy}(0,d,2d) + H_{efy}(0,d,0) = jw(2d)D_{efx}(0,d,d) \quad (24)$$

$$H_{efx}(d,0,2d) - H_{efx}(d,0,0) - H_{efz}(2d,0,d) + H_{efz}(0,0,d) = jw(2d)D_{efy}(d,0,d) \quad (25)$$

$$H_{ef\,x}(d,2d,0) - H_{ef\,x}(d,0,0) - H_{ef\,y}(2d,d,0) + H_{ef\,y}(0,d,0) = jw(2d)\,D_{ef\,z}(d,d,0) \quad (26)$$

Equations 12 and 13 have line integrals over the boundaries of z = +d face of unit cell and Equation 14 has surface integral *( ∬ ) ( ∬ )* over the face. Similar equations having line and surface integrals can be obtained for the x = +d and y = +d faces. Equations 6, 7 and 8 show relationship between the line integrals of electric field around the boundary of a given face of cubic cell and surface integral of magnetic field over the same face.

Using definitions of Equations 12, 13 and 14 to replace the components of Equations 9, 10 and 11 we get,

Till now, Equation 1 given by Maxwell has been used to find out the local fields and homogenized field expressions, but now Maxwell's Equation 2 will be used. According to Equation 1 circular electric field gives magnetic field and Equation 2 states that circular magnetic field results in electric field. Components of the Equation 2 are:

We obtain Equations 21, 22 and 23 when Equations 18, 19 and 20 are integrated over positive surfaces of x, y and z, respectively.

Homogenized field expressions[1] derived for Average Method by using general definitions of same are:

Field Energy Approach presents itself with the help of following definitions:

$$H_{efx}(d,2d,0)=\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_x(x,2d,0)\right|^2.\,dx} \qquad (27)$$

$$H_{efy}(2d,d,0)=\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_x(x,2d,0)\right|^2.\,dx} \qquad (28)$$

$$\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_x(x,2d,0)\right|^2dx}-\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_x(x,0,0)\right|^2dx}-\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_y(2d,y,+0)\right|^2dy}+\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_y(0,y,0)\right|^2dy}=jw(2d)\sqrt{\frac{1}{4d^2}\int_0^{2d}\int_0^{2d}\left|D(x,y,0)\right|^2dx.\,dy} \qquad (30)$$

$$\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_x(x,0,2d)\right|^2dx}-\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_x(x,0,0)\right|^2dx}-\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_z(2d,0,z)\right|^2dz}+\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_z(0,0,z)\right|^2dz}=jw(2d)\sqrt{\frac{1}{4d^2}\int_0^{2d}\int_0^{2d}\left|D(x,0,z)\right|^2dx.\,dz} \qquad (31)$$

$$\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_z(0,2d,z)\right|^2dz}-\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_z(0,0,z)\right|^2dz}-\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_y(0,y,2d)\right|^2dy}+\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_y(0,y,0)\right|^2dy}=jw(2d)\sqrt{\frac{1}{4d^2}\int_0^{2d}\int_0^{2d}\left|D(0,y,z)\right|^2dy.\,dz} \qquad (32)$$

$$D_{efz}(d,d,0)=\sqrt{\frac{1}{4d^2}\int_0^{2d}\int_0^{2d}\left|D_z(x,y,0)\right|^2dx.\,dy} \qquad (29)$$

Using definitions of Equations 27, 28 and 29 to replace the components of Equations 24, 25 and 26 we get,

Equations 30, 31 and 32 are the homogenized field expressions for Field Energy Approach which relate magnetic field line integral to the electric field surface integral.

To solve the equations and obtain parameters for empty unit cell, constitutive relation between fields E, D and B, H is used. But, since light entering metamaterial vary as $E_x(x, y, z) = E_0.\exp(jq_y y)$ and $H_z(x, y, z) = H_0.\exp(jq_y y)$, when y is direction of propagation of wave and $q_y$ is wave number, so above mentioned relation is also used apart from constitutive relation. We assume electric field to be in x and magnetic field in z -direction respectively. Using relation[6], $D = \epsilon E$

$$\epsilon_x=\frac{D_{efx}(0,d,d)}{E_{efx}(0,d,d)} \qquad (33)$$

Just like Equation 29 is defined from general definitions of Field Energy Method, we get expression for as,

$$D_{efx}(d,d,0)=\sqrt{\frac{1}{4d^2}\int_0^{2d}\int_0^{2d}\left|D_x(0,y,z)\right|^2dy.\,dz} \qquad (34)$$

Putting values of from Equation 34 and from Equation 12 we get,

$$\epsilon_x=\frac{\sqrt{\frac{1}{4d^2}\int_0^{2d}\int_0^{2d}\left|D_x(0,y,z)\right|^2dy.\,dz}}{\sqrt{\frac{1}{2d}\int_{-d}^{d}\left|E_x(x,+d,d)\right|^2.\,dx}} \qquad (35)$$

$$\epsilon_x=\epsilon_0 \qquad (36)$$

Similarly, calculation for permeability is done: Using relation[7], $B = \mu H$

$$\mu_{\square}=\frac{B_{efz}(0,0,d)}{H_{efz}(0,0,d)} \qquad (37)$$

Just like Equation 27 and 28 are defined from general definitions of Field Energy Method, we get expression for as,

$$H_{efz}(0,0,d)=\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_z(0,0,z)\right|^2.dz} \qquad (38)$$

Putting values of from Equation 14 and from Equation 38 we get,

$$\mu_z=\frac{\sqrt{\frac{1}{4d^2}\int_{-d}^{d}\int_{-d}^{d}\left|B_z(x,y,d)\right|^2dx.\,dy}}{\sqrt{\frac{1}{2d}\int_0^{2d}\left|H_z(0,0,z)\right|^2.\,dz}} \qquad (39)$$

If similar procedure is followed for unit cell having uniform medium, as mentioned above for empty unit then following expressions are obtained for and as,

$$€_x = €_0 . €_r \tag{40}$$

And,

$$\mu_z = \mu_0 . \mu_r \tag{41}$$

Here, $€_0$ is permittivity of free space and $€_r$ is relative permittivity. Similarly, $\mu_0$ permeability of free space and $\mu_r$ is relative permeability.

## 3. Analysis

For Average Method, a process is followed in which any function is taken as f(x) which can be Electric or Magnetic field. Then average of this function is obtained as:

$$f(x)_{avg} = \frac{1}{(b-a)} \int_a^b f(x).dx$$

From the above expression which we got for average function or field, energy is calculated for comparing it with the energy obtained in case of Field Energy Method.

$$\text{Energy} = \left|f(x)_{avg}\right|^2 = \frac{1}{(b-a)} \left|\int_a^b f(x).dx\right|^2 \tag{42}$$

Similarly, on following steps for Field Energy Method we get,

$$\text{Energy} = |f(x)|^2$$

$$\text{Total Energy} = \int_a^b |f(x)|^2 .dx$$

$$\text{Avg.(Total Energy)} = \frac{1}{(b-a)^2} \int_a^b |f(x).dx|^2 \tag{43}$$

From Equations 42 and 43, we can observe that energy in case of Field Energy Method is more and maintained than in case of Average Method.

Therefore, the expressions for permittivity (€) and permeability (μ) for an empty unit cell (with free space medium) of a metamaterial in Field Energy Method have been obtained correct and expected i.e. € and μ were equal to $€_0$ and $\mu_0$ respectively. Whereas in case of Average Method, the expressions of permittivity and permeability for an empty unit cell of a metamaterial were found to be approximate because of the extra sine term as shown below

$$€_x = \frac{€_0 . \sin(q_y d)}{q_y d}$$

$$\mu_z = \frac{\mu_0 . \sin(q_y d)}{q_y d}$$

In the similar way, correct and error free expressions have been obtained for a uniform medium as given by Equations 40 and 41.

## 4. Conclusion

In this work, the Field Energy Method is explored, than Average Method due to the fact that energy of the system is maintained in this method. Apart from this, results obtained for Field Energy Method are accurate whereas for Average Method, they are approximate. Initial work involves obtaining expressions for Field Energy Method to calculate parameters € and μ. The future work would include taking any example metamaterial and implementing the Energy Method for it.

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