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ORIGINAL ARTICLE

Mathematical modeling of COVID-19 in India and its states with optimal control

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Abstract
A pandemic is an epidemic spread over a huge geographical area. COVID-19 is 5th such pandemic documented after 1918 flu pandemic. In this work, we frame a mathematical epidemic model taking inspiration from the classic SIR model and develop a compartmental model with ten compartments to study the coronavirus dynamics in India and three of its most affected states, namely, Maharashtra, Karnataka, and Tamil Nadu, with inclusion of factors related to face mask efficacy, contact tracing, and testing along with quarantine and isolation. We fit the developed model and estimate optimum values of disease transmission rate, detection rate of undetected asymptomatic, and the same of undetected symptomatic. A sensitivity analysis is presented stressing on the importance of higher face mask usage, rapid testing, and contact tracing for curbing the disease spread. An optimal control analysis is performed with two control parameters to study the increase and decrease of the infected population with and without control. This study suggests that improved and rapid testing will help in identifying more infectives, thereby contributing in the decline of disease transmission rate. Optimal control analysis results on stressing on the importance of abiding by strict usage of face mask and social distancing for drastic decrease in number of infections. Time series behaviour of the symptomatic, asymptomatic, and hospitalized population is studied for a range of parameters, resulting in thorough understanding of significance of different parameters.

Keywords COVID-19 · Stability analysis · Sensitivity analysis · Optimal control · Testing and detection

Introduction

Novel Coronavirus, taxonomically termed as SARS-CoV-2, and COVID-19 by WHO, first emerged in Wuhan, Hubei Province in China Zhu et al. (2020) in late 2019. On 11th March 2020, this infectious disease was declared a pandemic by the World Health Organization. As on 13th February 2021, 107,838,255 confirmed cases have been reported worldwide, of which 2373,398 deaths have been reported by the https://covid19.who.int/. COVID-19 is the fifth documented pandemic since the 1918 flu pandemic. This deadly disease is highly contagious, transmitting from human to human, and spreading at an alarming rate. As per reports from https://www.who.int/health-topics/coronavirus, human-to-human transmission of this virus occurs through nasal discharge as well as saliva droplets when an infected person happens to sneeze or cough. This is also supported by the studies in Ferguson et al. (2020); Nicola et al. (2020), in which the novel coronavirus is compared to respiratory virus which spreads via airborne transmission. This virus is more likely to spread in regions which lack sanitation, ventilation Liu and Zhang (2020), and no usage of face shields. The virus can get inside body through contaminated hands that touch eyes, mouth, and nasal areas.

Control measure namely face mask and social distancing are considered to examine their impact on the dynamics of the disease, which is studied in detail in Okuonghae and Omame (2020). In Chu et al. (2020), the authors have done a metanalysis on the effectiveness of physical distancing, face mask, and eye protection in minimizing the human-to-human transmission of the disease. Both works by Eikenberry et al. (2020) and Ngonghala et al. (2020) stress on the importance of face mask or surgical masks in controlling the spread, and the later also suggests that with a higher efficacy of 70% or more, the disease could vanish. In Nadim and

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Chattopadhyay (2020), the authors have found results based on imperfect lockdown for countries like Mexico, Argentine, India, and South Africa. They determined that, in case of perfect lockdown, there would be a significant reduction in daily new COVID-19 cases. Apart from the above-mentioned interventions, it is equally important to have a well-functioning testing drive to tackle this deadly disease. Rapid testing and effective contact tracing help in identifying several asymptomatic and symptomatic cases, which will hence prevent disease transmission after being quarantined. In the paper by Ivorra et al. (2020), a new mathematical model θ -SEIHRD model based on Be-CoDis model in Ivorra et al. (2015) is proposed. This model considers existence of infectious undetected cases, control measures like sanitary conditions, isolation, quarantine, and tracing. It considers a novel method considering fraction of detected cases over actual total infected cases, which helps to understand the impact of COVID-19. In Zhang et al. (2021), it is clearly concluded that the spread of the disease could be controlled through effective contact tracing and by increasing the detection rates and quarantine of the individuals infectious to others.

In India, the first COVID-19 case was reported on January 30, 2020. To analyze the impact and transmission rate with and without interventions, scientific research towards combating this disease is very crucial to estimate the rise in cases, recoveries, and death. Several studies have been done since the mark of this disease and several works have been published worldwide. Mathematical modeling has played a crucial role in response to this pandemic, providing estimate on basic reproduction number across regions, analysis based on interventions included in the compartment models, quantifying disease severity, and so on. Most of the works are inspired by the compartment model SIR Kermack and McKendrick (1927), which is then extended to different epidemiological models for COVID-19. In Anirudh (2020), a brief study is done on prediction of COVID-19, its rise, spread, and reduction by giving description of different mathematical models, namely, SIR, SEIR, SEIRU,SIRD, SLIAR, ARIMA, SIDARTHE, etc., describing the different challenges and the outcomes obtained. In Meehan et al. (2020), authors have provided a brief history of this deadly pandemic, transmission of the disease with and without interventions and certain limitations associated with mathematical modeling. From these, it is clear that epidemiological models are important tools in public health management programmes, though high level of uncertainty persists in each of the models. The work by Saeed et al. (2021) provides a detailed information on various interventions depending on present-day pandemic situation.

In this study, we develop an epidemiological model with ten compartments, to study the coronavirus dynamics in India and three of its states. We have included natural births and deaths in the model. The major highlight of the model is that the population is categorized into four classes namely asymptomatic, symptomatic, detected, and undetected. Quarantine of the susceptible population, isolation, and hospitalization are included as well. The major aim is to determine how higher detection rates of asymptomatic and symptomatic unidentified individuals help in curbing the disease transmission rate. We have also depicted optimal control on this parameter by adding control, such that it enhances detection rate and hence reduces the number of unidentified infected individuals. In this work, we witness how face mask usage and quarantine rate of the susceptible help in reducing disease transmission, hence the basic reproduction number.

The rest of the paper is comprised of the following sections: "Model formulation: compartment model based on ordinary differential equations Sect. 2" includes a well-detailed explanation of the model formulation and the different movements taking place between compartments. The equilibria and the basic reproduction number are obtained in "Analysis of mathematical model Sect. 3" along with the local stability analysis of the disease-free and endemic equilibrium points. The next two sections are numerical simulation and optimal control of the problem. The final sections end with conclusion wherein the results and future scope are discussed in brief.

Model formulation: compartment model based on ordinary differential equations

In this study, the total population (N) is comprised of compartments, namely, Susceptible (S), Quarantined Susceptible (S_q), Exposed (E), Infected (I), Isolated/Quarantined (H_q), Hospitalized (H), and Recovered (R). The Infected compartment is further subdivided into four classes, namely, Asymptomatic Infected Undetected (I_{an}), Detected (I_{ad}) and Symptomatic Infected Undetected (I_{sn}) and Detected (I_{sd}). In this work, the major focus is on the undetected and detected population of the infected, since interventions related to these are included for optimal control of the system. Figure 1 is a schematic representation of entire model flow.

Considering the below-mentioned assumptions, the mathematical model is developed.

1. In this model, we assume that the individuals are recruited at a constant rate $\wedge$ in a specified region.
2. By means of contact tracing, it is assumed that the susceptible individuals are identified and quarantined at a movement rate of ν, and m_q is the rate at which quarantined individuals move to susceptible class.
3. β is the infection rate at which the susceptible move to the exposed class. A certain fraction of individuals wear face masks and abide by the norms, for which we

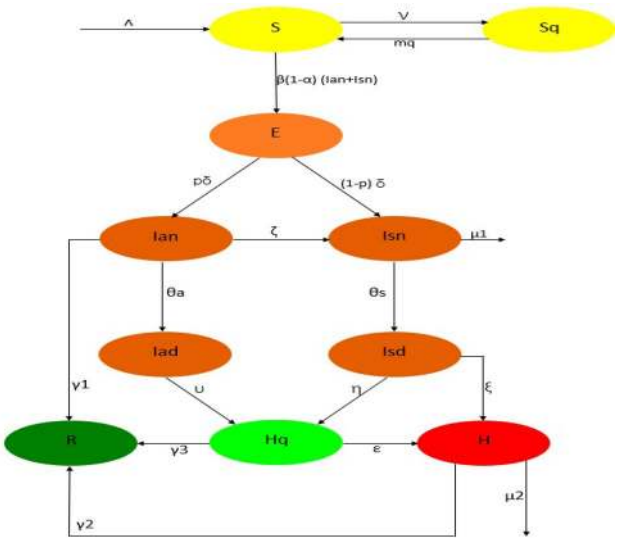

Fig. 1 Schematic diagram of the epidemic model

include a parameter α which is product of mask efficacy and fraction of population is wearing it. Hence, the rate of disease transmission becomes $\beta(1-\alpha)$.

4. $p\delta$ and $(1-p)\delta$ are the rate of movement of individuals from exposed class to I_{an} and I_{sn} class respectively.
5. The asymptomatic undetected individuals recover at rate γ_1. Asymptomatic individuals have possibilities of becoming symptomatic, and hence, they move from I_{an} to I_{sn} at a rate ζ.
6. There are chances of the symptomatic undetected individuals to be not identified hence leading to death, and therefore, we assume that these individuals die at a rate μ_1.
7. θ_a and θ_s are the rates of detection via contact tracing and testing for I_{an} and I_{sn} class, respectively.
8. The asymptomatic detected individuals I_{ad} are isolated at a movement rate of υ to the class H_q.
9. The symptomatic detected individuals I_{sd} who happen to be critically ill are hospitalized at a rate ξ. There are possibilities of the symptomatic detected individuals to not have severe symptoms, and hence, they move to the class H_q at a rate η.
10. There is a possibility for the individuals under isolation to develop complication, and hence, they move to H class at a rate ϵ.
11. The recovery rates of isolated and hospitalized are γ_3 and γ_2 respectively. μ_2 is the death rate of hospitalized individuals. Let μ be the natural death rate of the population N.

In view of the above, the following mathematical model is developed:

$$\frac{dS}{dt} = \wedge - \nu S - \beta(I_{an} + I_{sn})S(1-\alpha) + m_q S_q - \mu S, \tag{1}$$

$$\frac{dS_q}{dt} = \nu S - m_q S_q - \mu S_q, \tag{2}$$

$$\frac{dE}{dt} = \beta(I_{an} + I_{sn})S(1-\alpha) - \delta E - \mu E, \tag{3}$$

$$\frac{dI_{an}}{dt} = p\delta E - \gamma_1 I_{an} - \zeta I_{an} - \theta_a I_{an} - \mu I_{an}, \tag{4}$$

$$\frac{dI_{sn}}{dt} = (1-p)\delta E - \mu_1 I_{sn} - \theta_s I_{sn} + \zeta I_{an} - \mu I_{sn}, \tag{5}$$

$$\frac{dI_{ad}}{dt} = \theta_a I_{an} - \upsilon I_{ad} - \mu I_{ad}, \tag{6}$$

$$\frac{dI_{sd}}{dt} = \theta_s I_{sn} - \xi I_{sd} - \eta I_{sd} - \mu I_{sd}, \tag{7}$$

$$\frac{dH_q}{dt} = \upsilon I_{ad} - \gamma_3 H_q + \eta I_{sd} - \epsilon H_q - \mu H_q, \tag{8}$$

$$\frac{dH}{dt} = \xi I_{sd} - \gamma_2 H - \mu_2 H + \epsilon H_q - \mu H, \tag{9}$$

$$\frac{dR}{dt} = \gamma_1 I_{an} + \gamma_2 H + \gamma_3 H_q - \mu R. \tag{10}$$

Analysis of mathematical model

Positivity and boundedness

The solution set of the system of Eqs. (1)–(10) are bounded by $\frac{\wedge}{\mu}$. Hence, biologically feasible region for the system of Eqs. (1)–(10) is given by

$$\Omega_2 = \Big\{(S, S_q, E, I_{an}, I_{sn}, I_{ad}, I_{sd}, H_q, H, R) \in \mathbb{R}_+^{10} : 0 \leq S, S_q, E, I_{an}, I_{sn}, I_{ad}, I_{sd}, H_q, H, R \leq \frac{\wedge}{\mu}\Big\}.$$

Equilibrium and basic reproduction number

The disease-free equilibrium of the model represented by Eqs. (1)–(10) is given by

$$E_0 = \left(\frac{\wedge(m_q+\mu)}{\mu(m_q+\nu+\mu)}, \frac{\wedge\nu}{\mu(m_q+\nu+\mu)}, 0,0,0,0,0,0,0,0\right).$$

Using the Next Generation Matrix Method as in van den Driessche and Watmough (2002), Diekmann et al. (1990), Hethcote (2000), we obtain the expression for basic reproduction number (R_0). Basic reproduction number is defined as the average number of secondary cases produced by a single infected case in an otherwise susceptible population. We compute

$$\mathcal{F} = \begin{pmatrix} \beta(1-\alpha)(I_{an}+I_{sn})S \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathcal{V} = \begin{pmatrix} (\delta+\mu)E \\ -p\delta E + (\gamma_1+\zeta+\theta_a+\mu)I_{an} \\ -(1-p)\delta E - \zeta I_{an} + (\mu_1+\theta_s+\mu)I_{sn} \\ -\theta_a I_{an} + (\upsilon+\mu)I_{ad} \\ -\theta_s I_{sn} + (\xi+\eta+\mu)I_{sd} \end{pmatrix}.$$

These two matrices depict the new infections and transition terms, respectively. The next two matrices F and V are Jacobian of $\mathcal{F}$ and $\mathcal{V}$, respectively

$$F = \begin{pmatrix} 0 & \beta(1-\alpha)\frac{\wedge(m_q+\mu)}{\mu(m_q+\nu+\mu)} & \beta(1-\alpha)\frac{\wedge(m_q+\mu)}{\mu(m_q+\nu+\mu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} \delta+\mu & 0 & 0 & 0 & 0 \\ -p\delta & \gamma_1+\zeta+\theta_a+\mu & 0 & 0 & 0 \\ -(1-p)\delta & -\zeta & \mu_1+\theta_s+\mu & 0 & 0 \\ 0 & -\theta_a & 0 & \upsilon+\mu & 0 \\ 0 & 0 & -\theta_s & 0 & \xi+\eta+\mu \end{pmatrix}.$$

The basic reproduction number (R_0) is the largest eigenvalue of FV^{-1}, given by

$$R_0 = \frac{\wedge}{\mu}\frac{\beta(1-\alpha)(m_q+\mu)\delta}{(m_q+\mu+\nu)(\delta+\mu)} \times \frac{\big((1-p)(\gamma_1+\zeta+\theta_a+\mu)+p\zeta+p(\mu_1+\theta_s+\mu)\big)}{(\gamma_1+\zeta+\theta_a+\mu)(\mu_1+\theta_s+\mu)}.$$

An unique Endemic Equilibrium (EE) of the system (1)–(10) exists provided $R_0 > 1$ and is given by

$$EE = (S*, S_q*, E*, I_{an}*, I_{sn}*, I_{ad}*, I_{sd}*, H_q*, H*, R*),$$

where

$$R^* = \frac{1}{\mu}\left(\gamma_1 I^*_{an} + \gamma_2 H^* + \gamma_3 H^*_q\right),$$

$$H^* = \frac{1}{d_6}\left(\xi I^*_{sd} + \epsilon H^*_q\right),$$

$$H^*_q = \frac{1}{d_5}\left(\upsilon I^*_{ad} + \eta I^*_{sd}\right),$$

$$I^*_{sd} = \frac{\theta_s}{d_4} I^*_{sn},$$

$$I^*_{ad} = \frac{\theta_a}{(\upsilon+\mu)} I^*_{an},$$

$$I^*_{sn} = \frac{(\zeta I^*_{an} + (1-p)\delta E^*)}{d_2},$$

$$I^*_{an} = \frac{p\delta E^*}{d_1}, \quad S^*_q = \frac{\nu S^*}{d_3},$$

$$E^* = \frac{1}{\delta+\mu}\left(\wedge - \frac{\mu(m_q+\mu+\nu)}{(m_q+\mu)}S^*\right),$$

$$S^* = \frac{(\delta+\mu)d_1 d_2}{\beta(1-\alpha)\delta(pd_2+p\zeta+(1-p)d_1)},$$

where, $d_1 = \gamma_1+\zeta+\theta_a+\mu; d_2 = \mu_1+\theta_s+\mu; d_3 = m_q+\mu;$ $d_4 = \xi+\eta+\mu; d_5 = \gamma_3+\epsilon+\mu; d_6 = \gamma_2+\mu_2+\mu.$

Stability analysis

In this section, we state the theorems on local stability of the equilibria. The proofs are omitted as they are trivial.

Theorem 1 *The Disease-free Equilibrium given by E_0 is locally asymptotically stable under certain restrictions when $R_0 < 1$ and is unstable otherwise. The restrictions are*

$$(\mu+\delta+d_1+d_2)(d_1d_2+(d_1+d_2)(\delta+\mu)) > (\delta+\mu)d_1d_2 - S^0\beta\delta(1-\alpha)(1-d_1) + S^0\beta\delta p(1-\alpha)(d_1-d_2-\zeta),$$

where $S^0 = \frac{\wedge(m_q+\mu)}{\mu(m_q+\nu+\mu)}$, $d_1 = \gamma_1+\zeta+\theta_a+\mu$ *and* $d_2 = \mu_1+\theta_s+\mu$.

Theorem 2 *The Endemic Equilibrium given by EE which exists if $R_0 > 1$ is locally asymptotically stable under certain restrictions and is unstable otherwise. The restrictions are*

$$A_1 > 0, \quad A_1A_2 - A_3 > 0,$$
$$A_1A_2A_3 + A_1A_5 - A_1^2A_4 - A_3^2 > 0$$
$$A_1A_2A_3A_4 + 2A_1A_5A_4 + A_3^2A_5 - A_3^2A_4 - A_5^2 - A_1^2A_4^2 - A_1A_2^2A_5 > 0$$
$$A_1A_2A_3A_4A_5 + 2A_1A_4A_5^2 + A_2A_3A_5^2 - A_1^2A_4^2A_5 - A_1A_2^2A_5^2 - A_3^2A_4A_5 - A_5^3 > 0,$$

where

Table 1 Values of parameters for the model (1)

Parameter		Value	References
$\wedge$	:	Varies	
ν	:	0.8958	Sarkar et al. (2020)
α	:	0.05	Ngonghala et al. (2020); Davies et al. (2013)
p	:	[0.15,0.7]	Ferguson et al. (2020); Li et al. (2020)
m_q	:	0.0417	Tang et al. (2012)
δ	:	[0.071,0.33]	Lauer et al. (2020); Li et al. (2020)
γ_1	:	.12	Assumed
γ_2	:	1/14	Zhou et al. (2020); Tang et al. (2020)
γ_3	:	1/7.48	Sarkar et al. (2020)
ζ	:	[0.01,0.08]	Aldila et al. (2020)
μ_1	:	0.0002	Assumed
μ_2	:	0.00002	Assumed
υ	:	0.2	Ngonghala et al. (2020)
ξ	:	[0.02,0.1]	Ferguson et al. (2020); Li et al. (2020)
η	:	0.07151	Tang et al. (2020)
ϵ	:	0.02	Assumed
μ	:	0.000425	Demographic

$$
\begin{aligned}
A_1 &= (d_1 + d_2 + d3 + \delta + \mu + j_{11}) \\
A_2 &= d_1d_2 + d_1d_3 + d_2d_3 + (d_1 + d_2 + d_3)(\delta + \mu + j_{11}) \\
&\quad + \delta(j_{11} - j_{14}) + \mu(j_{11} - m_q) \\
A_3 &= d_1d_2d_3 + (d_1d_2 + d_1d_3 + d_2d_3) \\
&\quad (\delta + \mu + j_{11}) + j_{11}(d_1 + d_2 + d_3)(\delta + \mu) \\
&\quad - (d_1 + d_2 + \delta + \mu)m_q\nu \\
&\quad + \delta j_{14}(d_1p - d_2p - p\zeta - (\mu + \nu)) \\
A_4 &= \delta j_{14}\nu(m_q - d_3 - d_1) - \big(d_1d_2 + (d_1 + d_2)(\delta + \mu)\big)m_q\nu \\
&\quad + j_{11}(\delta + \mu)(d_1d_2 + d_1d_3) \\
&\quad + j_{11}(\delta + \mu)(d_2d_3) + (d_1d_2d_3)(\mu + \delta + j_{11}) \\
&\quad + \delta j_{14}(d_1 - d_2)p(d_3 + \mu + \nu) \\
&\quad - \delta j_{14}\big(d_1d_3 + (d_1 + d_3)\mu \\
&\quad + p\zeta(d_3 + \mu + \nu)\big) \\
A_5 &= d_1d_2d_3j_{11}(\delta + \mu) - d_1d_3\delta j_{14}(\nu + \mu) \\
&\quad - d_1\delta m_q\nu(d_2 - j_{14}) - d_1d_2m_q\nu\mu \\
&\quad + \delta j_{14}p\big((d_1 - d_2)(d_3\mu \\
&\quad + d_3\nu - m_q\nu) - \zeta(d_3(\mu + \nu) - m_q\nu)\big),
\end{aligned}
$$

where $j_{11} = (\nu + \mu) + \frac{\beta(1-\alpha)\delta}{d_1d_2}\frac{\wedge}{(\delta+\mu)}\big(d_2p + (1-p)d_1 + \zeta p\big)$, $j_{14} = \frac{(\delta+\mu)d_1d_2}{\delta(pd_2+(1-p)d_1+\zeta p)}$.

Numerical simulation

Data and model fitting

Best fit of data with optimum parameter values help us in identifying those key parameters which will help in controlling the disease spread. In this work for model fitting and optimal parameter estimation, we have worked with active COVID-19 cases data of India and its three states, namely Maharashtra, Karnataka, and Tamil Nadu. The data are collected from https://www.covid19india.org/. The collected data are of 8 months starting from May 1, 2020 to December 31, 2020. From this model calibration, we have estimated the

Table 2 Estimated optimum parameter values and basic reproduction number

States	Estimated parameter values	Estimated R_0 value
Maharashtra	$\beta = 6.935 \times 10^{-}6$	$R_0 = 1.8727$
	$\theta_a = .056$	
	$\theta_s = .073$	
Karnataka	$\beta = 4.006 \times 10^{-}6$	$R_0 = 1.5803$
	$\theta_a = .218$	
	$\theta_s = .048$	
Tamil Nadu	$\beta = 2.338 \times 10^{-}6$	$R_0 = 1.1032$
	$\theta_a = .063$	
	$\theta_s = .047$	
India	$\beta = 2.065 \times 10^{-}7$	$R_0 = 1.1146$
	$\theta_a = .095$	
	$\theta_s = .037$	

optimum values of three parameters which are disease transmission rate (infection rate), detection rate of asymptomatic undetected individuals, and detection rate of symptomatic undetected individuals. The remaining parameter values are listed in Table 1.

Numerical simulation of the model by means of data fitting is done using R software. We have used the sum of least square method (Betti and Heffernan 2021) to estimate β, θ_a and θ_s which best fit the observed active cases for all the four data sets. These estimated parameter values along with the value of R_0 are mentioned in Table 2.

Considering the three states for comparison, we see that R_0 and β values are maximum for Maharashtra, followed by Karnataka and least is for Tamil Nadu. In a similar way, the disease transmission rate or infection rate (β) is highest for Maharashtra and least for Tamil Nadu. This can be related with the total confirmed cases in these three states (https://www.covid19india.org/). Since we have included parameter related to detection in the developed model, we obtain optimum values of θ_a and θ_s for which we get the best fit. We observe that the total detection rate is maximum in Karnataka, and as of December 31, 2020 a total of 14,078,158 testing were completed (https://www.covid19india.org/). The total detection rates of Maharashtra and Tamil Nadu differ by 0.019. Higher detection rate implies reduction in the disease transmission rate, since more infected individuals get detected, and their chances of coming in contact with disease-free individuals are nullified, hence drastically reducing the infection rate.

The higher value of R_0 in Maharashtra suggests that the spread of the disease is faster, which can be justified with the fact that being a state with 120 million plus population, the testing stood at only 14 million individuals. Hence, this can be due to no proper testing or contact tracing. The other reasons could be no compulsory usage of face mask and no practice of social distancing. The model fitting is depicted in Fig. 2, wherein the simulations are done for 243 days. In all the four figures (a), (b), (c), and (d), the curve is bending for longer duration of time, implying the cases will reduce in number and saturate at the last data set.

Figure 3 depicts the prediction for 100 more days, and the curve is a decreasing curve implying reduction in the COVID cases, which adheres to the present-day COVID-19 scenario.

Sensitivity analysis

Sensitivity analysis is crucial in determining the importance of various parameters in disease transmission. In Rodrigues et al. (2013), a detailed explanation on sensitivity analysis for case of dengue is presented. It helps in determining the model robustness, as different predictions are made, which are exposed to chances of error, be it in data collection or different assumed parameter values. Sensitivity analysis helps in identifying the parameters with high and low impact on the reproduction number, thereby helping in focusing on various intervention strategies. The normalized forward sensitivity index of a variable with respect to a parameter is the ratio of the relative change in the variable to the relative change in the parameter. When the variable is a differentiable function of the parameter, the sensitivity index may be alternatively defined using partial derivatives. From Chitnis et al. (2008), the normalized forward sensitivity index of R_0, that depends differentiably on a parameter a, is defined by

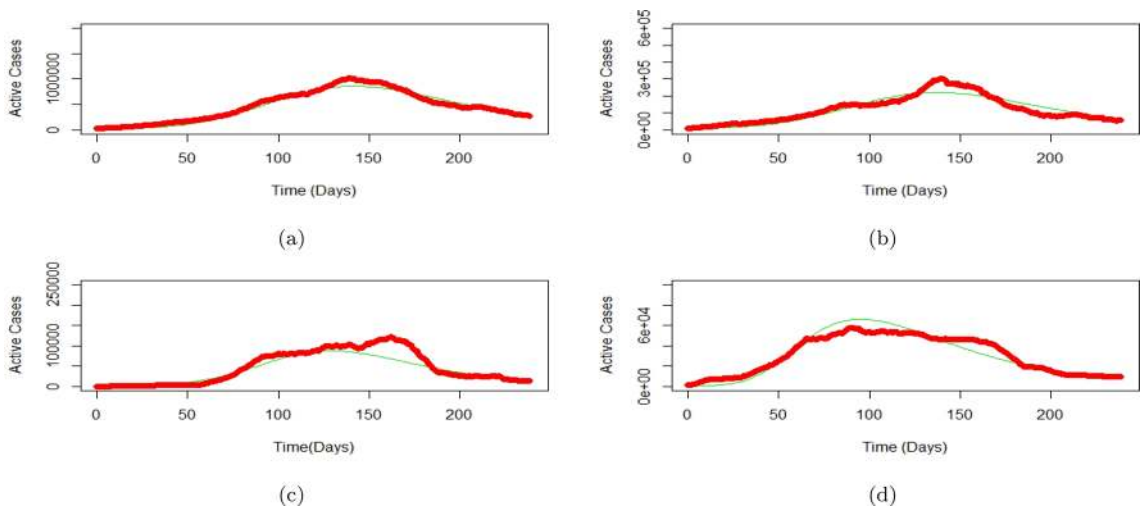

Fig. 2 Plots of the fitted mode with observed COVID-19 cases for **a** India, **b** Maharashtra, **c** Karnataka, and **d** Tamil Nadu. The red dots represent observed data and the green curve is the model solution

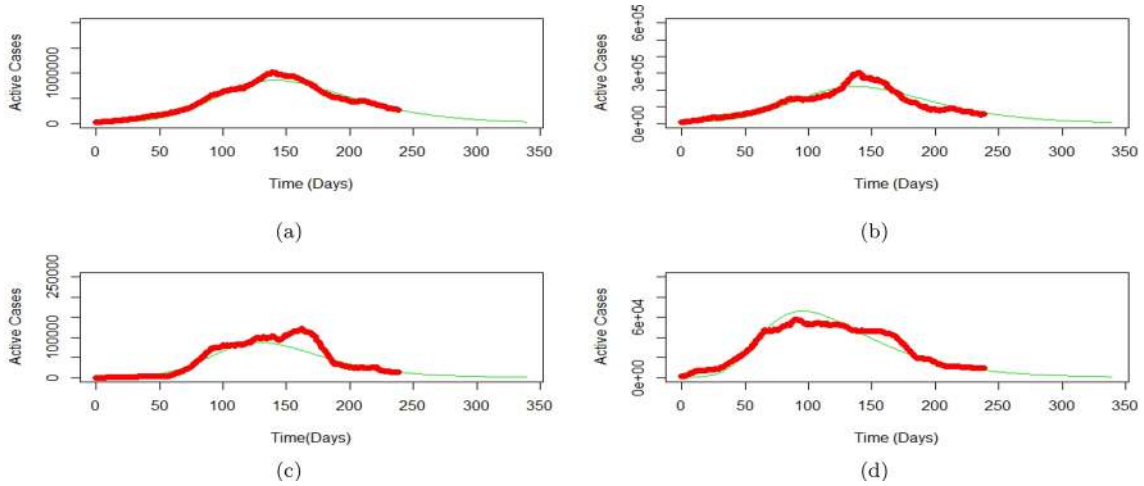

Fig. 3 Plots of the fitted mode with observed COVID-19 cases with predictions for **a** India, **b** Maharashtra, **c** Karnataka, and **d** Tamil Nadu. The red dots represent observed data and the green curve is the model solution

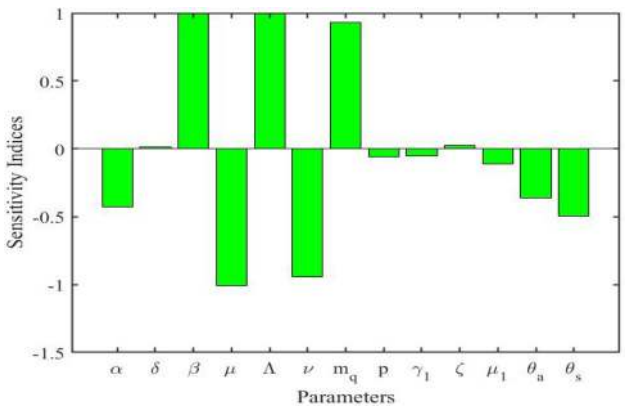

Fig. 4 Normalized forward sensitivity indices of R_0 with respect to the model parameters $\wedge = 500$, $\nu = 0.8958$, $\alpha = 0.3$, $m_q = 0.0417$, $\delta = 0.071$, $\gamma_1 = 0.12$, $\gamma_2 = 0.0714$, $\gamma_3 = 0.137$, $\zeta = 0.06$, $\mu_1 = 0.0002$, $\mu_1 = 0.0002$, $\mu = 0.000425$, $p = 0.6995$, $\beta = 0.0000028$, $\theta_a = 0.063$, $\theta_s = 0.047$, $\upsilon = 0.2$, $\xi = 0.09$, $\eta = 0.0715$, $\epsilon = 0.02$

$$r_a^{R_0} = \frac{\partial R_0}{\partial a} \times \frac{a}{R_0}.$$

Figure 4 depicts the normalized forward sensitivity indices of R_0 with respect to various parameters. The figure clearly denotes that the parameters β, δ, m_q, ζ, and $\wedge$ have positive indices with R_0. This suggests that with increase in the values of these parameters, the R_0 value increases. Hence, these parameters are required to be controlled, so as to control the spread of the disease. From the figure, we note that the sensitivity indices of β and $\wedge$ are 1, which means that R_0 value increases by 1 percent if these parameters are increased by 1 percent. Similarly, the parameters α, μ, ν, p, γ_1, μ_1, θ_a, and θ_s share negative indices with the basic reproduction number. We note that increases in the values of parameters namely detection rates and face mask efficacy, as well as increase in quarantine rate of susceptible have major impact in reduction of the basic reproduction number.

Figures 5, 6, and 7 depict how certain parameters can bring down the value of basic reproduction number below 1 and hence the epidemic potential. The contour plot (Fig. 5) shows that increase in the recovery rate of the asymptomatic individuals will result in reduction of R_0 as well as lower disease transmission rates will bring down the R_0 value at significant rate. This could be sufficed with the fact that if more number of unidentified asymptomatic individuals have higher recovery rate, then chances of transmission of infection from these individuals are quite less. From the contour plot (Fig. 6), we conclude that as both mask efficacy and fraction of people wearing it, increases along with a higher rate at which the susceptible are quarantined, the epidemic potential reduces to value below unity. The reason here is, if more people are adhering to compulsory mask usage, the lesser will be the transmission of disease from infected to susceptible individuals. Similarly, if more number of individuals are quarantined at initial stage, there are chances of earlier detection. From Fig. 7, it can be concluded that greater the detection rate, lower will be the risk of disease transmission. This is due to the reasoning that, once the individuals are identified to be infectious, they will be either quarantined, isolated, or hospitalized. Hence, the probability of passing of infection from these individuals is reduced significantly.

Impact of different parameters on prevalence of COVID-19

In this section, we present the time series analysis of the model (1)–(10) for the undetected asymptomatic,

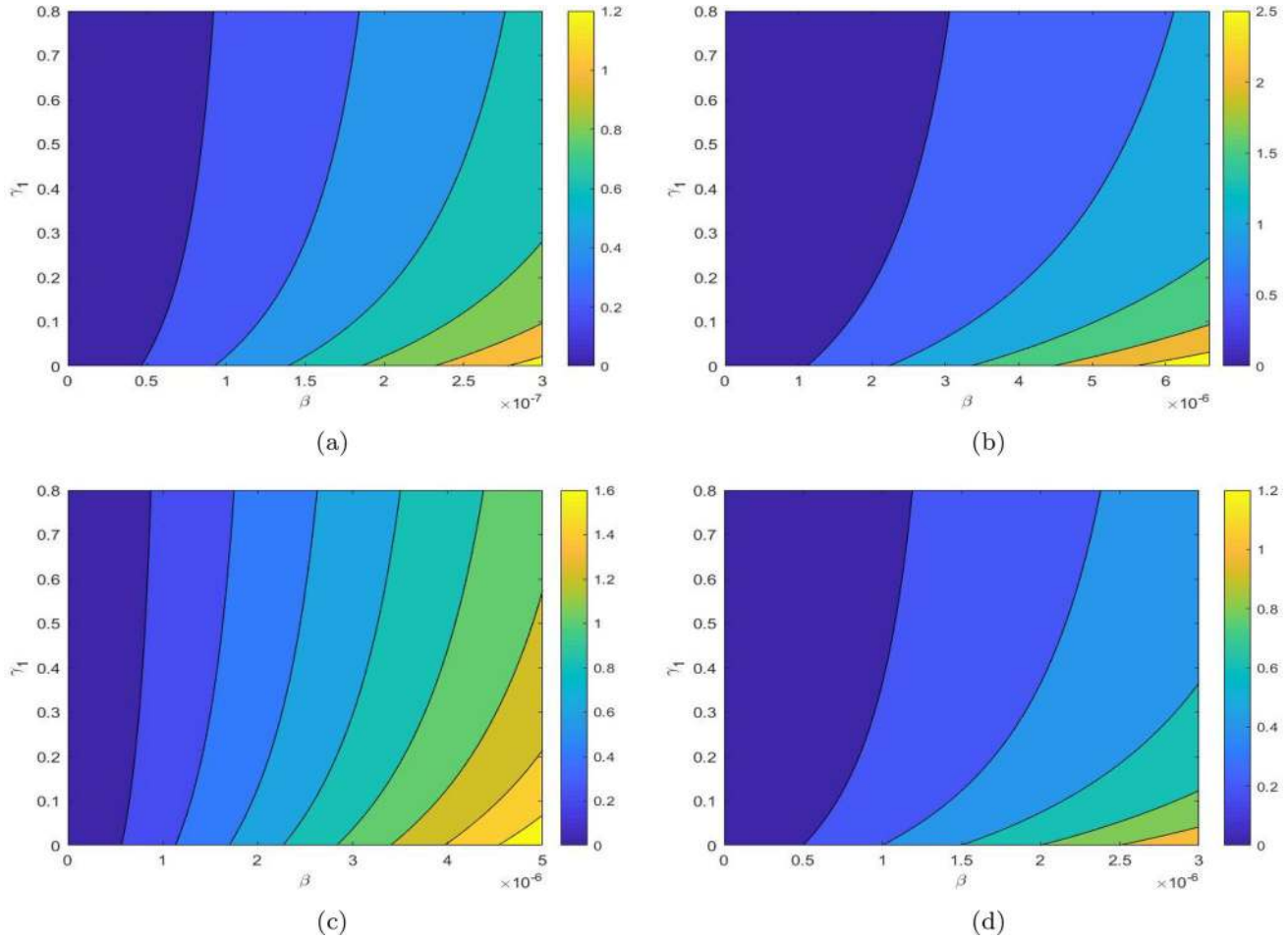

Fig. 5 Contour plot of the basic reproduction number with respect to the infection rate (β) and recovery rate of asymptomatic undetected (γ_1) for **a** India, **b** Maharashtra, **c** Karnataka, and **d** Tamil Nadu. Parameter values: **a** $\theta_a = .095$, $\theta_s = .037$; **b** $\theta_a = .056$, $\theta_s = .073$; **c** $\theta_a = .218$, $\theta_s = .048$; **d** $\theta_a = .063$, $\theta_s = .047$. The rest of the parameters are as in Table 1

symptomatic, and hospitalized population for Maharashtra, Karnataka, Tamil Nadu, and India as a whole. We study the behaviour of these curves by changing parameter values. Figure 8 shows that with increase in the recovery rates of I_{an}, the number of individuals belonging to this class will reduce. This happens as with lesser days of recovery, the possibilities of infectious pathogens getting transmitted is less. A similar behaviour is observed in Fig. 9, where the variation in I_{an} is studied with respect to detection rate of asymptomatic individuals. Here, with the rise in detection rates by means of say rapid testing kits, home testing tools, well functioning of contact tracing, etc. results in movement of these individuals to detected classes. Therefore, we get decreasing curves with increasing θ_a value. Similar behaviour is observed in the case of symptomatic unidentified class (I_{sn}) in Fig. 10. Figure 11 depicts decrease in the hospitalized population with increase in the recovery rate. Similar pattern is observed for all three states and the country as a whole.

Optimal control

Optimal control problem

In this section, we extend the mathematical model, which is presented by the system of Eqs. (1)–(10) by adding control parameters for the formulation of optimal control problem. u_1 and u_2 are the two control parameters included in the model. Here, u_1 represents the control parameter which is responsible for reducing the transmission rate (β). This control parameter can be equated with compulsory use of face mask, sanitation, and hand gloves usage. The other control u_2, helps in detecting the asymptomatic and symptomatic unidentified population, thereby reducing the population in these compartments. This is possible, since rapid testing, contact tracing, and enhanced home testing tool kits will result in over all improvement in the detection rate. $\theta_a + u_2$ and $\theta_s + u_2$ are the enhanced detection rate with rapid testing, home testing tools, and contact tracing. These two control functions are bounded and Lebesgue

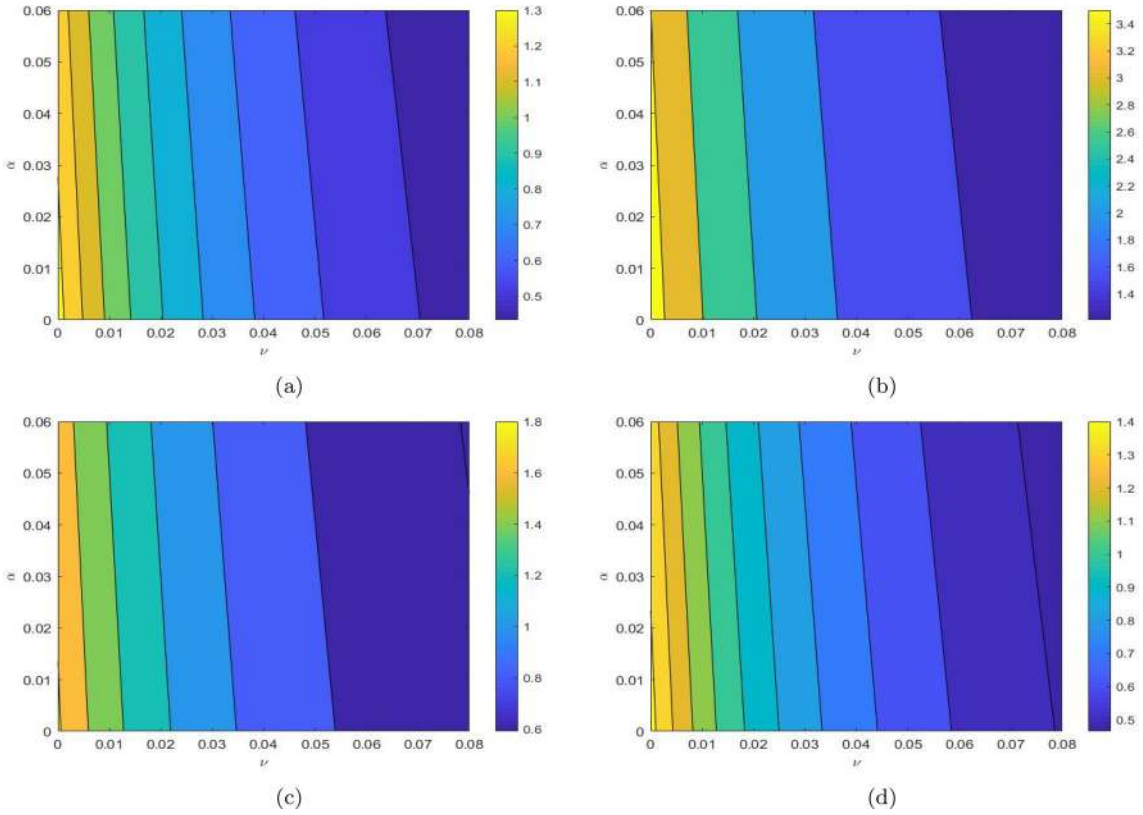

Fig. 6 Contour plot of the basic reproduction number with respect to the face mask efficacy (α) and quarantine rate of susceptibles (ν) for **a** India, **b** Maharashtra, **c** Karnataka, and **d** Tamil Nadu. Parameter values: **a** $\beta = 2.065 \times 10^{-}7$, $\theta_a = .095$, $\theta_s = .037$; **b** $\beta = 6.935 \times 10^{-}6$, $\theta_a = .056$, $\theta_s = .073$; **c** $\beta = 4.006 \times 10^{-}6$, $\theta_a = .218$, $\theta_s = .048$; **d** $\beta = 2.338 \times 10^{-6}$, $\theta_a = .063$, $\theta_s = .047$. The rest of the parameters are as in Table 1

integrable on $[0, t_f]$, where t_f is the pre-fixed time interval length to which controls are applied. It is assumed that u_1 and u_2 lie between 0 and 1, since if both equal zero, it implies no efforts are placed in these controls. Similarly, maximum effort relates to these values being 1. With the above assumptions, the following optimal control model is formulated:

$$\begin{aligned}\frac{dS}{dt} &= \wedge - \nu S - (1-\alpha)(1-u_1(t))\beta S(I_{an} + I_{sn}) \\ &\quad + m_q S_q - \mu S,\end{aligned} \tag{11}$$

$$\frac{dS_q}{dt} = \nu S - m_q S_q - \mu S_q, \tag{12}$$

$$\frac{dE}{dt} = (1-\alpha)(1-u_1(t))\beta S(I_{an} + I_{sn}) - \delta E - \mu E, \tag{13}$$

$$\frac{dI_{an}}{dt} = p\delta E - \gamma_1 I_{an} - \zeta I_{an} - (\theta_a + u_2(t))I_{an} - \mu I_{an}, \tag{14}$$

$$\begin{aligned}\frac{dI_{sn}}{dt} &= (1-p)\delta E - \mu_1 I_{sn} - (\theta_s + u_2(t))I_{sn} \\ &\quad + \zeta I_{an} - \mu I_{sn},\end{aligned} \tag{15}$$

$$\frac{dI_{ad}}{dt} = (\theta_a + u_2(t))I_{an} - \upsilon I_{ad} - \mu I_{ad}, \tag{16}$$

$$\frac{dI_{sd}}{dt} = (\theta_s + u_2(t))I_{sn} - \xi I_{sd} - \eta I_{sd} - \mu I_{sd}, \tag{17}$$

$$\frac{dH_q}{dt} = \upsilon I_{ad} - \gamma_3 H_q + \eta I_{sd} - \epsilon H_q - \mu H_q, \tag{18}$$

$$\frac{dH}{dt} = \xi I_{sd} - \gamma_2 H - \mu_2 H + \epsilon H_q - \mu H, \tag{19}$$

$$\frac{dR}{dt} = \gamma_1 I_{an} + \gamma_2 H + \gamma_3 H_q - \mu R. \tag{20}$$

The objective functional for the fixed t_f is given by

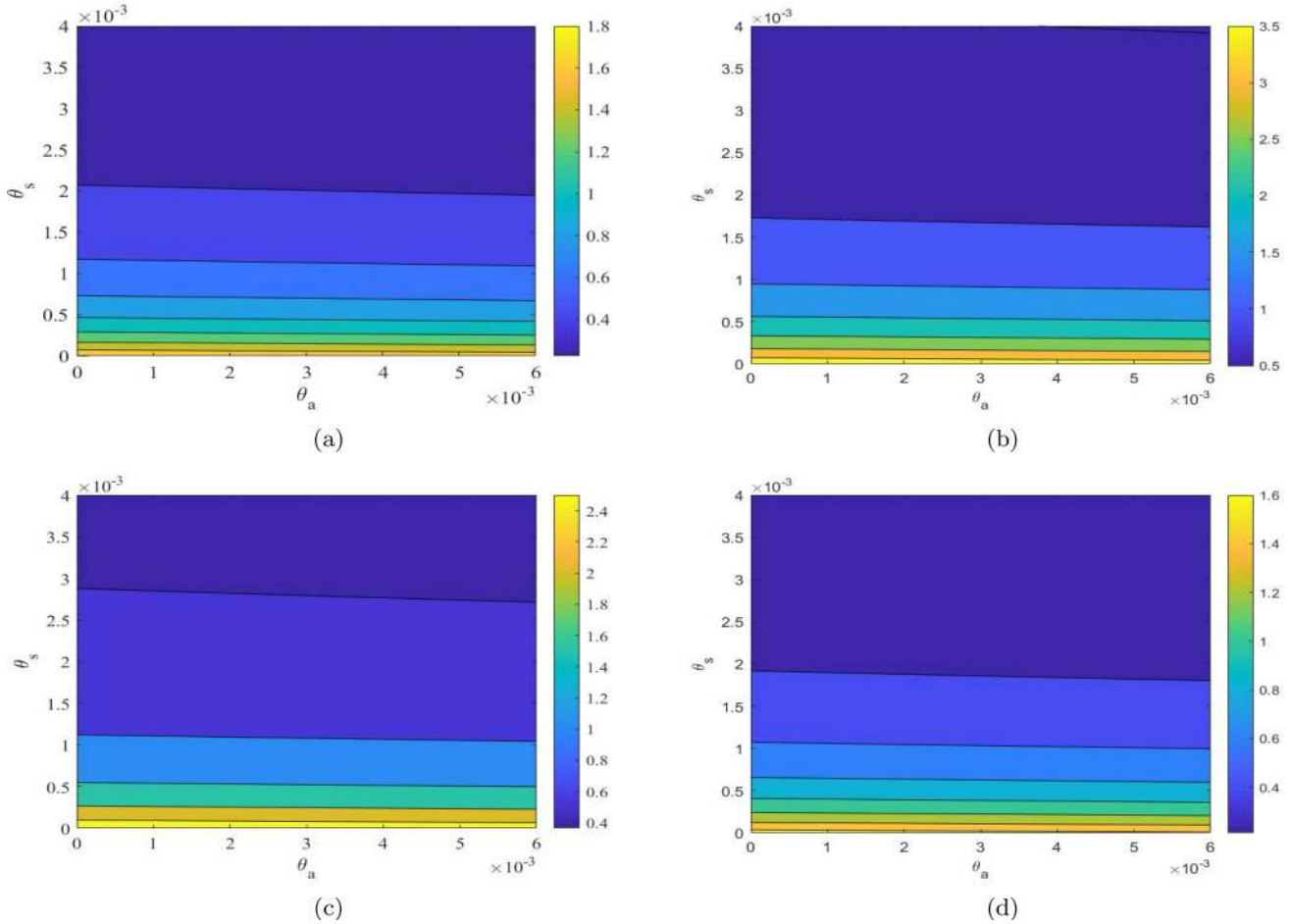

Fig. 7 Contour plot of the basic reproduction number with respect to the asymptomatic detection rate (θ_a) and detection rate of symptomatic (θ_s) for **a** India, **b** Maharashtra, **c** Karnataka, and **d** Tamil Nadu. Parameter values: **a** $\beta = 2.065 \times 10^{-7}$, **b** $\beta = 6.935 \times 10^{-6}$, **c** $\beta = 4.006 \times 10^{-6}$, **d** $\beta = 2.338 \times 10^{-6}$. The rest of the parameters are as in Table 1

$$J = \int_0^{t_f} \left(C_1 I_{an} + C_2 I_{sn} + C_3 I_{ad} + C_4 I_{sd} + \frac{1}{2} C_5 u_1^2 + \frac{1}{2} C_6 u_2^2\right) dt, \tag{21}$$

where $C_1, C_2, C_3, C_4, C_5, C_6 \geq 0$ are the weight constants.

Objective is to find the control parameters $u_1 *$, $u_2 *$, such that

$$J(u_1 *, u_2 *) = \min_{u_1, u_2 \in \Omega_1} J(u_1, u_2),$$

where Ω_1 is the control set, defined as

$$\Omega_1 = \left\{u_1, u_2 : measurable \quad and \quad 0 \leq u_1, u_2 < 1\right\} \quad and \quad t \in [0, t_f].$$

The Lagrangian of this problem is

$$L(I_{an}, I_{sn}, I_{ad}, I_{sd}, u_1, u_2) = C_1 I_{an} + C_2 I_{sn} + C_3 I_{ad} + C_4 I_{sd} + \frac{1}{2} C_5 u_1^2 + \frac{1}{2} C_6 u_2^2.$$

The Hamiltonian $\mathcal{H}$ formed for our problem is

$$\mathcal{H} = L(I_{an}, I_{sn}, I_{ad}, I_{sd}, u_1, u_2) + \lambda_1 \frac{dS}{dt} + \lambda_2 \frac{dS_q}{dt} + \lambda_3 \frac{dE}{dt} + \lambda_4 \frac{dI_{an}}{dt} + \lambda_5 \frac{dI_{sn}}{dt} + \lambda_6 \frac{dI_{ad}}{dt} + \lambda_7 \frac{dI_{sd}}{dt} + \lambda_8 \frac{dH_q}{dt} + \lambda_9 \frac{dH}{dt} + \lambda_{10} \frac{R}{dt},$$

where $\lambda_i' s$ are the adjoint variables ($i = 1$–10).

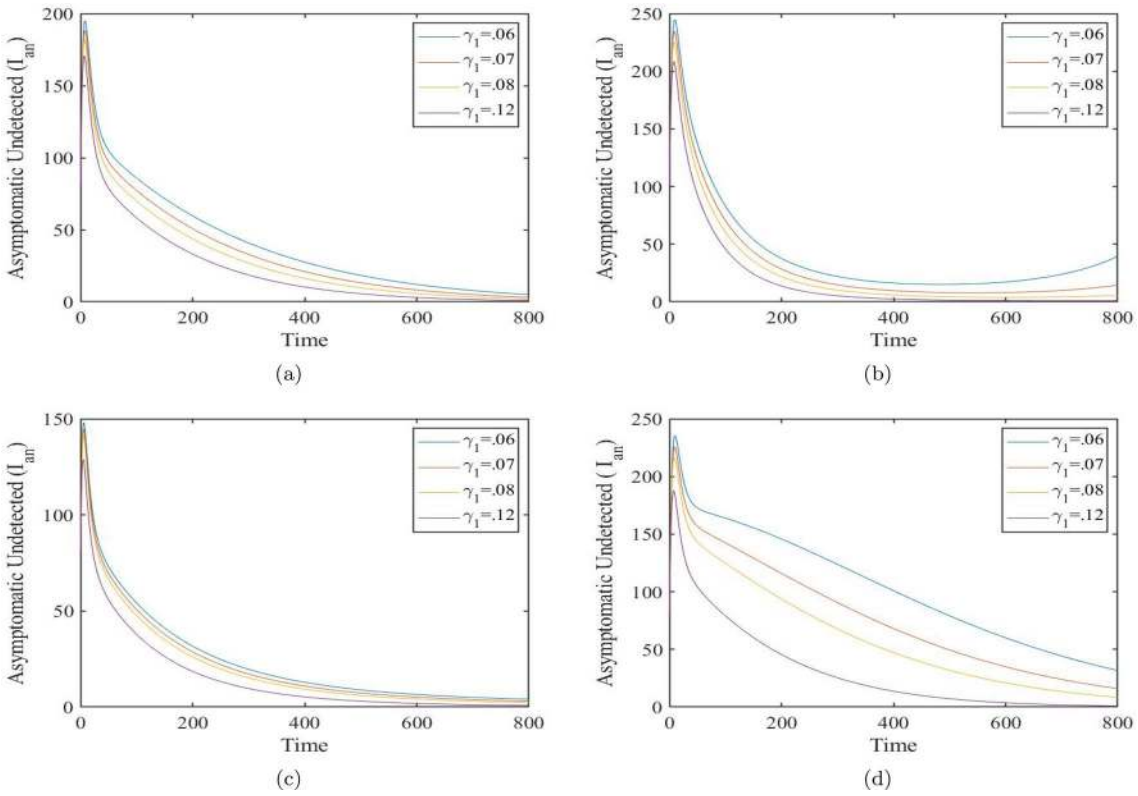

Fig. 8 Time series of model (1) showing variations in asymptomatic undetected individuals I_{an} with respect to γ_1 for **a** India, **b** Maharashtra, **c** Karnataka, and **d** Tamil Nadu. Parameters are same as Fig. 6

The adjoint variables are written in the form of differential equations as follows:

$$\frac{d\lambda_1}{dt} = -\frac{\partial \mathcal{H}}{\partial S} = \lambda_1\mu + (\lambda_1 - \lambda_2)\nu + (\lambda_1 - \lambda_3)\beta(1-\alpha)(1-u_1(t))(I_{an} + I_{sn}), \tag{22}$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial \mathcal{H}}{\partial S_q} = (\lambda_2 - \lambda_1)m_q + \lambda_2\mu, \tag{23}$$

$$\frac{d\lambda_3}{dt} = -\frac{\partial \mathcal{H}}{\partial E} = (\lambda_3 - \lambda_4)p\delta + (\lambda_3 - \lambda_5)(1-p)\delta + \lambda_3\mu, \tag{24}$$

$$\frac{d\lambda_4}{dt} = -\frac{\partial \mathcal{H}}{\partial I_{an}} = -C_1 + (\lambda_1 - \lambda_3)\beta(1-\alpha)(1-u_1(t))S + (\lambda_4 - \lambda_{10})\gamma_1 + (\lambda_4 - \lambda_5)\zeta + (\lambda_4 - \lambda_6)(\theta_a + u_2(t)) + \lambda_4\mu, \tag{25}$$

$$\frac{d\lambda_5}{dt} = -\frac{\partial \mathcal{H}}{\partial I_{sn}} = -C_2 + (\lambda_1 - \lambda_3)\beta(1-\alpha)(1-u_1(t))S + (\lambda_5 - \lambda_7)(\theta_s + u_2(t)) + \lambda_5(\mu_1 + \mu), \tag{26}$$

$$\frac{d\lambda_6}{dt} = -\frac{\partial \mathcal{H}}{\partial I_{ad}} = -C_3 + (\lambda_6 - \lambda_8)\upsilon + \lambda_6\mu, \tag{27}$$

$$\frac{d\lambda_7}{dt} = -\frac{\partial \mathcal{H}}{\partial I_{sd}} = -C_4 + (\lambda_7 - \lambda_9)\xi + (\lambda_7 - \lambda_8)\eta + \lambda_7\mu, \tag{28}$$

$$\frac{d\lambda_8}{dt} = -\frac{\partial \mathcal{H}}{\partial H_q} = (\lambda_8 - \lambda_{10})\gamma_3 + (\lambda_8 - \lambda_9)\epsilon + \lambda_8\mu, \tag{29}$$

$$\frac{d\lambda_9}{dt} = -\frac{\partial \mathcal{H}}{\partial H} = (\lambda_9 - \lambda_{10})\gamma_2 + \lambda_9(\mu_2 + \mu), \tag{30}$$

$$\frac{d\lambda_{10}}{dt} = -\frac{\partial \mathcal{H}}{\partial R} = \lambda_{10}\mu. \tag{31}$$

Let $\widetilde{S}, \widetilde{S_q}, \widetilde{E}, \widetilde{I_{an}}, \widetilde{I_{sn}}, \widetilde{I_{ad}}, \widetilde{I_{sd}}, \widetilde{H_q}, \widetilde{H}$, and $\widetilde{R}$ be optimum values of $S, S_q, E, I_{an}, I_{sn}, I_{ad}, I_{sd}, H_q, H$, and R, respectively. Let $\widetilde{\lambda_1}, \widetilde{\lambda_2}, \widetilde{\lambda_3}, \widetilde{\lambda_4}, \widetilde{\lambda_5}, \widetilde{\lambda_6}, \widetilde{\lambda_7}, \widetilde{\lambda_8}, \widetilde{\lambda_9}$, and $\widetilde{\lambda_{10}}$ be solution of (8). Using (Pontryagin et al. 1962; Pontryagin 1987; Lenhart and Optimal 2007), we state and prove the below theorem.

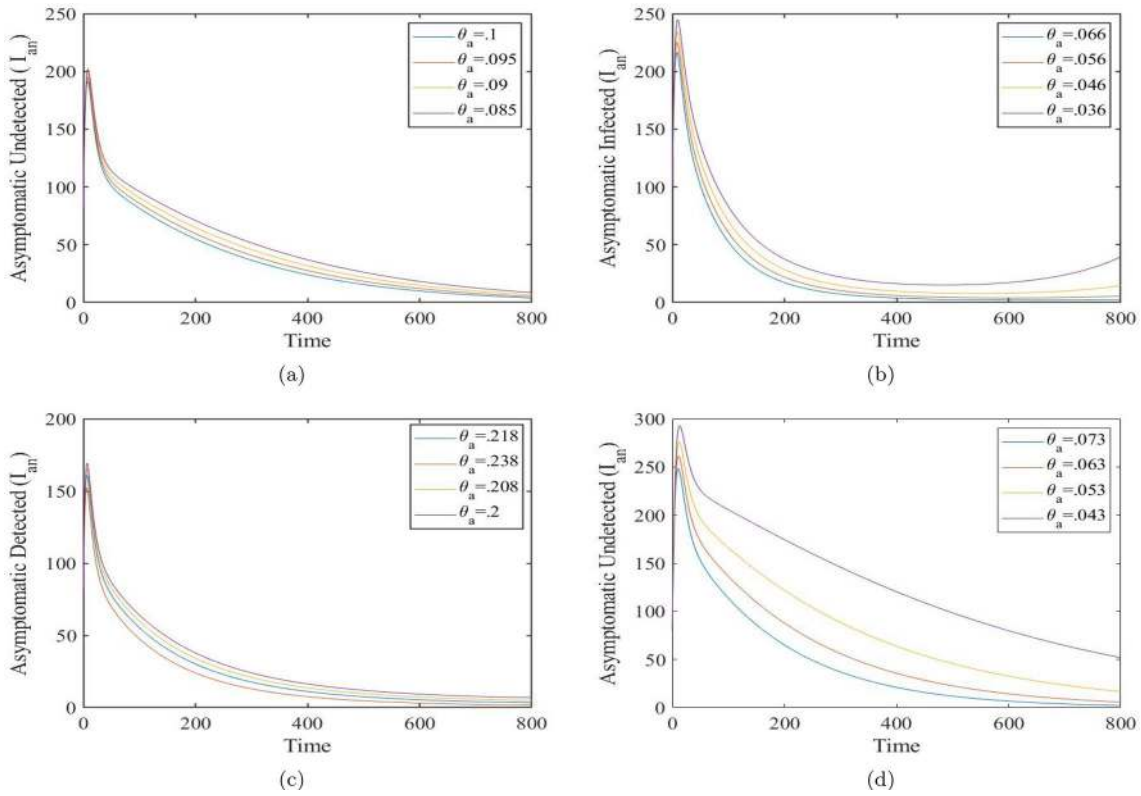

Fig. 9 Time series of model (1) showing variations in asymptomatic undetected individuals I_{an} with respect to θ_a for **a** India, **b** Maharashtra, **c** Karnataka, **d** Tamil Nadu. Parameters are same as Fig. 6

Theorem 3 *There exists optimal controls* $u_1 *, u_2 * \in \Omega_1$, *such that* $J(u_1 *, u_2 *) = \min J(u_1, u_2)$ *subject to extended system of Eqs.* (11)–(20) .

Proof We use (Pontryagin et al. 1962) to prove this theorem. In this case, we observe that the controls are non-negative. The necessary convexity of the objective functional in (u_1, u_2) is satisfied for minimizing the problem. The set of control variable $u_1, u_2 \in \Omega_1$ is convex and closed by definition. The state variables are bounded and the integrand of the functional $C_1 I_{an} + C_2 I_{sn} + C_3 I_{ad} + C_4 I_{sd} + \frac{1}{2} C_5 u_1^2 + \frac{1}{2} C_6 u_2^2$ is convex on Ω_1. Since there exist optimal controls for minimizing the functional subject to systems (11)–(20) and (22)–(31), we use Pontryagin's maximum principle (Pontryagin et al. 1962) to derive the necessary conditions to find the optimal solutions in the following way:

Suppose (z, u) is an optimal solution of an optimal control problem, then this implies that there exist a non-trivial vector function $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$ satisfying the following:

$$\frac{dz}{dt} = \frac{\partial \mathcal{H}(t, z, u, \lambda)}{\partial \lambda}, \quad 0 = \frac{\partial \mathcal{H}(t, z, u, \lambda)}{\partial \lambda},$$
$$\frac{d\lambda}{dt} = \frac{\partial \mathcal{H}(t, z, u, \lambda)}{\partial \lambda}.$$

□

Theorem 4 *The optimal controls* $u_1 *, u_2 *$ *which minimize* J *over the region* Ω_1 *are given by*

$$u_1 *= \min\left\{1, \max(0, \widetilde{u_1}\right\}$$
$$u_2 *= \min\left\{1, \max(0, \widetilde{u_2}\right\},$$

where

$$\widetilde{u_1} = \frac{(\lambda_3 - \lambda_1)\beta(1-\alpha)S(I_{an} + I_{sn})}{C_5}$$
$$\widetilde{u_2} = \frac{(\lambda_4 - \lambda_6)I_{an} + (\lambda_5 - \lambda_7)I_{sn}}{C_6}.$$

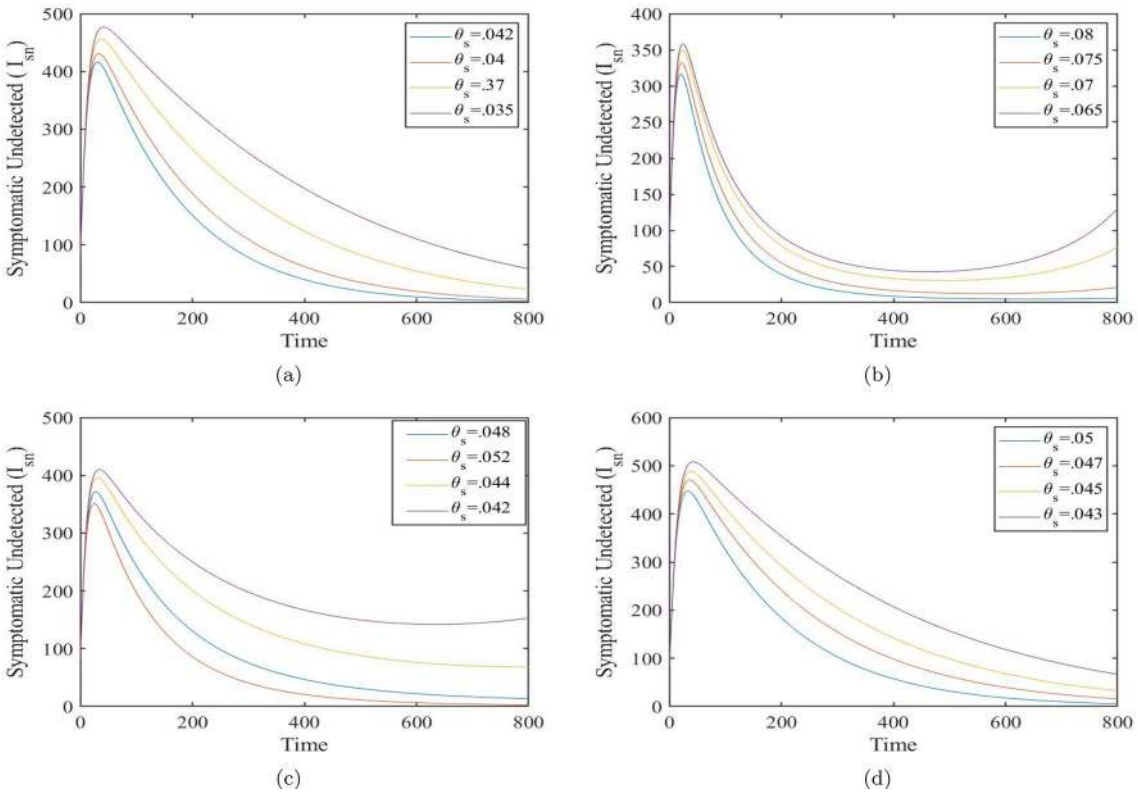

Fig. 10 Time series of model (1) showing variations in symptomatic undetected individuals I_{sn} with respect to θ_s for **a** India, **b** Maharashtra, **c** Karnataka, and **d** Tamil Nadu. Parameters are same as Fig. 6

Proof We prove this theorem using (Pontryagin 1987; Pontryagin et al. 1962) and Theorem 3.

Using the optimally condition: $\frac{\partial \mathcal{H}}{\partial u_1} = 0, \frac{\partial \mathcal{H}}{\partial u_2} = 0$, we get

$$\frac{\partial \mathcal{H}}{\partial u_1} = C_5 u_1 + (\lambda_1 - \lambda_3)\beta(1-\alpha)S(I_{an} + I_{sn}) = 0$$

$$\implies u_1 = \frac{(\lambda_3 - \lambda_1)\beta(1-\alpha)S(I_{an} + I_{sn})}{C_5} = \widetilde{u_1}$$

and $$\frac{\partial \mathcal{H}}{\partial u_2} = C_6 u_2 + (\lambda_6 - \lambda_4)I_{an} + (\lambda_7 - \lambda_5)I_{sn}$$

$$\implies u_2 = \frac{(\lambda_4 - \lambda_6)I_{an} + (\lambda_5 - \lambda_7)I_{sn}}{C_6} = \widetilde{u_2}.$$

Again, the lower bound is 0 and upper bound is 1 for the controls u_1 and u_2. This suggests that $u_1 = u_2 = 0$ if $\widetilde{u_1} < 0$ and $\widetilde{u_2} < 0$, also $u_1 = u_2 = 1$ if $\widetilde{u_1} > 1$ and $\widetilde{u_2} > 1$, otherwise $u_1 = \widetilde{u_1}$ and $u_2 = \widetilde{u_2}$. Therefore, for these controls $u_1 *$ and $u_2 *$, we get optimum values of J. □

Optimal control model simulation

The simulation of the optimal control problem is done in MATLAB by using the set of parameter values which corresponds to stability of endemic equilibrium (EE). The time interval is taken to be 150 days. The weight constants are $C_1 = 1, C_2 = 1, C_3 = 1, C_4 = 1, C_5 = 90, C_6 = 100$. The extended system of Eqs. (11)–(20) is solved by iterative method using forward and backward difference approximation. The control profiles of u_1 and u_2 are depicted in Fig. 12. Figure 13 shows the variation in undetected asymptomatic and symptomatic population with and without control. It is very evident that inclusion of control parameters will reduce the cases of I_{an} and I_{sn}, since enhanced detection and reduced infection rate will reduce the undetected infected population. Hence, optimal control is effective in decreasing the infective population within a desired interval of time.

Discussion

The numerical simulations provided different results and a clear interpretation on the pandemic situation for the considered time period. The curve for infected cases is predicted for next 100 days and it did abide by the pandemic trend. In the Sect. "Numerical Simulation", we get the optimum estimates for infection rate and detection rate of the three states and India. Among the three states, Maharashtra has an highest infection rate of 6.935×10^{-6} and thereby larger Reproduction

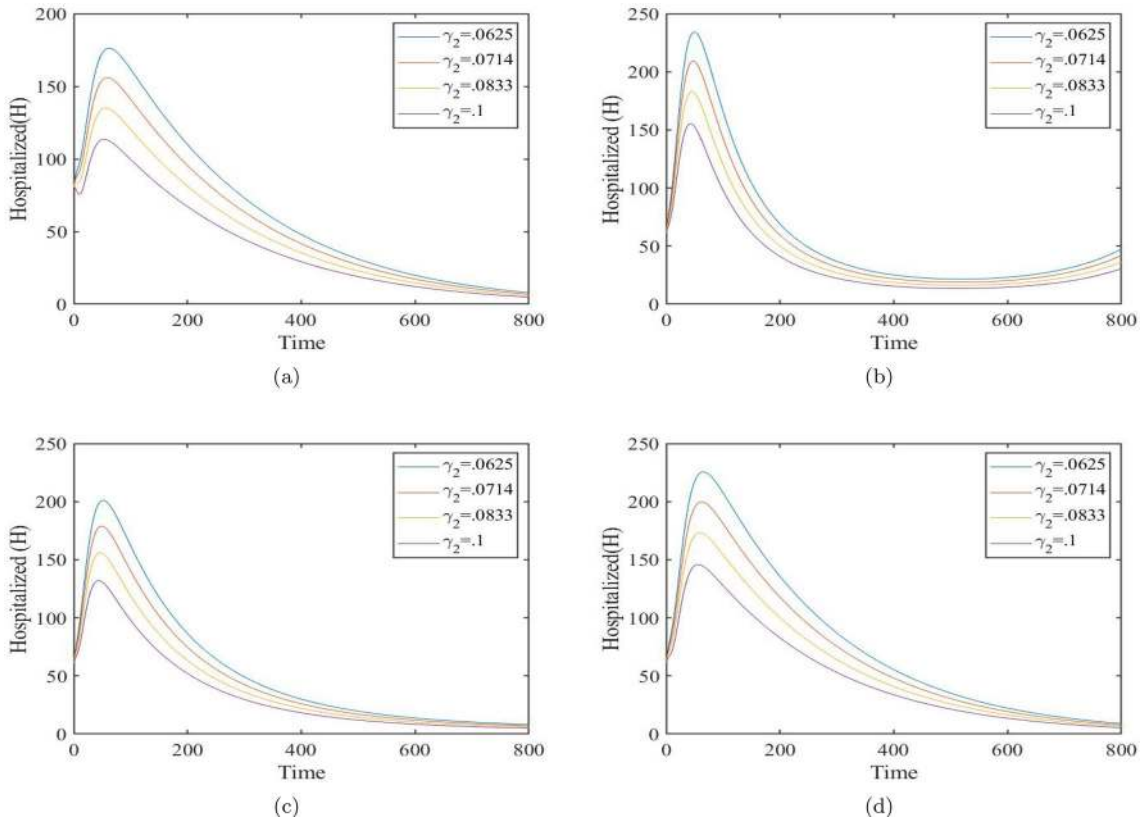

Fig. 11 Time series of model (1) showing variations in hospitalized individuals H with respect to γ_2 for **a** India, **b** Maharashtra, **c** Karnataka, and **d** Tamil Nadu. Parameters are same as Fig. 6

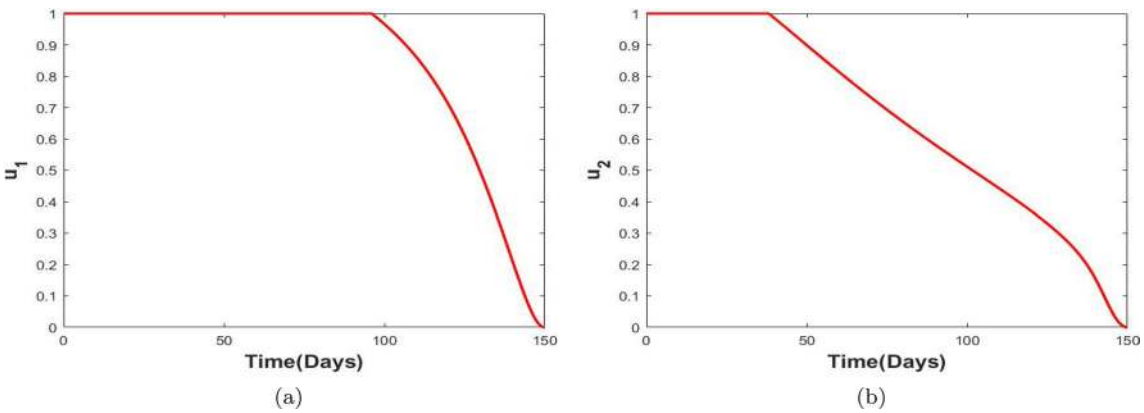

Fig. 12 Control profile of **a** u_1 and **b** u_2

Number value of 1.8727. This indicates that the R_0 value will keep rising as the number infectives increases rapidly. The sensitivity analysis performed in Sect. "Numerical Simulation" suggests the same. Along with the infection rate, the significance of other parameters is also determined under sensitivity analysis. The lowest infection rate among the three states was observed for the state of Tamil Nadu being 2.338×10^{-6}. This satisfies with the data of the state, as it reported less number of total active cases in comparison with other two states. Tamil Nadu witnessed more number of recoveries in comparison as per (https://www.covid19india.org/). Under optimal control analysis in Sect. "Optimal Control", increase in the detection of infectives and rapid testing help in curbing the disease transmission. Compulsory mask usage, relating to control parameter u_1 and rapid testing, contact tracing relating to control parameter u_2 will contribute in bringing down the infection

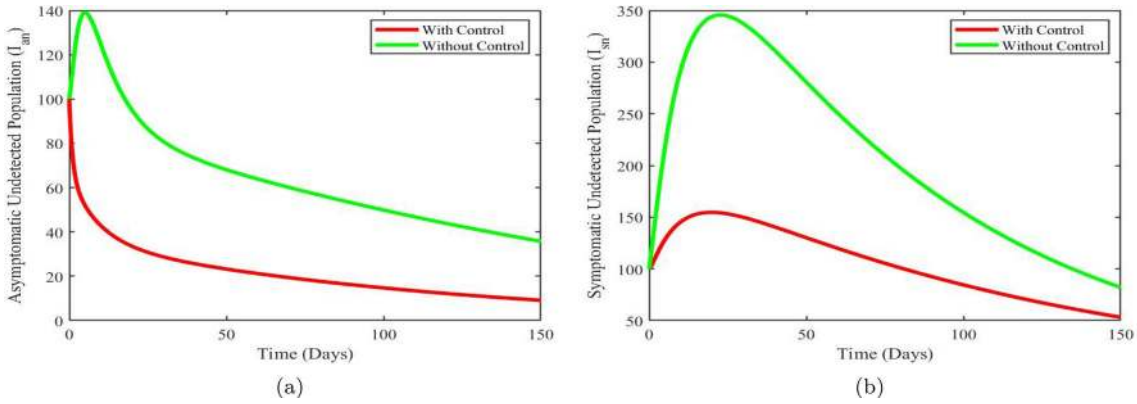

Fig. 13 Variation of **a** I_{an} and **b** I_{sn} with and without control

rate by a larger margin. If these interventions are not followed, the infections will spike up. Hence, these results adhere to the significance of parameters involved in developing the model.

Conclusion

Epidemic models help to understand the way in which the spread of disease takes place. These compartmental mathematical models play a significant role in determining various control parameters by which new policies and interventions can be implemented to bring in a decline in the growing trend of disease spread. In this study, we developed an epidemic model with ten compartments and performed mathematical analysis on it. We obtained two unique equilibrium point namely disease-free equilibrium and endemic equilibrium, and proved local stability under certain restrictions on parameters. The study is based on 8 months data of COVID-19 cases in India and three of its states. The model fitting is done for the four data sets by means of least square method in R software, by which we obtained optimum parameter values of disease transmission rate, and detection rate of undetected asymptomatic and symptomatic population for the best fit of the collected data. From the results of parameter estimation, we note that the infection rate is maximum for Maharashtra and the detection rate is maximum for Karnataka that agree with the reported data from https://www.covid19india.org/. We fitted the model with predictions, wherein we observed that the curve keeps decreasing in the next 100 days, which agrees with the actual data trend as in https://www.covid19india.org/. Sensitivity analysis is performed, which gives a detailed explanation of each parameter and its impact on the reproduction number. We witness that the higher values of detection rates and face mask efficacy result in decline of basic reproduction number. It also shows that early quarantine and higher quarantine rates of the susceptible help in reducing the number of unidentified infected population. All these contribute in reduction in arrival of secondary cases. Time series behaviour is obtained to study the variations in asymptomatic, symptomatic, and hospitalized compartment with rise and fall of parameter values. In this study, we have extended the model to optimal control problem by incorporating two control parameters, one to reduce disease transmission and the other to enhance the detection rate. From all these, we conclude that if policies related to rapid testing, monitored contact tracing, compulsory mask, and gloves usage are implemented, then the spread of this deadly pandemic can be controlled. The model in future could be extended by taking several other parameters related to age and inclusion of population with cases of respiratory ailments. This will help to get a deeper perspective on the infection spread and a well-categorized result based on the assumptions made.

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