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Turán type inequalities for q-hypergeometric functions

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Abstract

In this paper our aim is to deduce some Turán type inequalities for q-hypergeometric and q-confluent hypergeometric functions. In order to obtain the main results we apply the methods developed in the case of classical Kummer and Gauss hypergeometric functions.

Keywords: Basic hypergeometric functions; q-Kummer confluent hypergeometric functions; Gauss and Kummer hypergeometric functions; Arithmetic, geometric mean; Turán type inequalities

1. Introduction

S. Karlin and G. Szegő [10] investigated a general theory dealing with inequalities of the type

$$P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \geq 0, \tag{1.1}$$

where $\{P_n\}$ are the classical orthogonal polynomials. In particular, when $P_n(x)$ is the Legendre polynomial and $|x| \leq 1$, (1.1) is the well-known Turán inequality [17]. Extensive study of this quadratic form and its analogues was carried out in various directions. For example, results similar to (1.1) in the direction of general orthogonal polynomials were studied in [5–16]. For

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more literature on Turán inequality for various orthogonal polynomials and special functions, we refer the reader to the details given in [1,2] and references therein. Since Turán's inequality was investigated for the orthogonal polynomials having hypergeometric representation, it is worth studying the validity of such inequality for various hypergeometric functions as well. In [4] Turán type inequalities for Kummer's confluent hypergeometric function were discussed, which complement the results from [3]. In this paper, we would like to present the q-version of some results obtained in [1,2,4] for the classical Gauss and Kummer hypergeometric functions.

The q-hypergeometric series is given by Olver et al. [15, p. 423]

$$ {}_2\phi_1(a, b, c; q, x) = \sum_{n\geq 0} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} x^n, \tag{1.2}$$

where $0 < q < 1$, $|x| < 1$, $a, b, c \in \mathbb{R}$ such that c does not take any of the values q^{-n}, and

$$ (a;q)_0 = 1; \qquad (a;q)_n = \prod_{m=0}^{n-1} (1 - aq^m), \quad n \in \mathbb{N};$$
$$ (a;q)_\infty = \lim_{n\to\infty} (a;q)_n = \prod_{m\geq 0} (1 - aq^m)$$

is the q-shifted factorial. Note that for $q \nearrow 1$ the expression $(q^a;q)_n/(1-q)^n$ tends to $(a)_n = a(a+1)\cdots(a+n-1)$, and thus the basic hypergeometric series reduces to the well-known Gaussian hypergeometric function. More precisely, we have

$$ \lim_{q\nearrow 1} {}_2\phi_1(q^a, q^b, q^c; q, x) = {}_2F_1(a, b, c; x) = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n n!} x^n,$$

where ${}_2F_1$ stands for the Gaussian hypergeometric function [15, p. 384]. For later use let us consider also the q-gamma function, defined by Olver et al. [15, p. 145]

$$ \Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}.$$

It should be mentioned here that $x \mapsto \log \Gamma_q(x)$ is convex on $(0, \infty)$, which can be proved easily by using the series representation of the q-digamma function, that is,

$$ \psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)} = -\log(1-q) + (\log q) \sum_{m\geq 1} \frac{q^{mx}}{1-q^m}.$$

Indeed, we have

$$ \psi_q'(x) = \left(\frac{\Gamma_q'(x)}{\Gamma_q(x)} \right)' = (\log q)^2 \sum_{m\geq 1} \frac{m q^{mx}}{1-q^m},$$

which shows that ψ_q is increasing on $(0, \infty)$, or in other words Γ_q is log-convex on $(0, \infty)$. For more details on the q-gamma and q-digamma functions we refer to [6,8,9,13,14] and to references therein.

This paper is organized as follows: in Section 2 we present a Turán type inequality for a particular case of the q-hypergeometric function, while in Section 3 we investigate some Turán type inequalities for the q-Kummer confluent hypergeometric function.

2. Turán type inequalities for the q-hypergeometric function

In this section, we consider the q-hypergeometric function

$$\phi_a(x) := {}_2\phi_1(q^a, q^{c-a}, q^c; q, x) = \sum_{n\geq 0} \frac{(q^a;q)_n (q^{c-a};q)_n}{(q^c;q)_n (q;q)_n} x^n \tag{2.1}$$

and we extend the main results from [1,2] to this function. To this aim we need the following preliminary result.

Lemma 2.1. *Let n be a natural number, $0 < q < 1$, $0 \leq a \leq c \leq 1$, and consider the sequences of functions $\{f_n\}_{n\geq 1}$, $\{g_n\}_{n\geq 1}$ and $\{h_n\}_{n\geq 1}$, defined by $f_n(a) = (q^a;q)_n(q^{c-a};q)_n$, $g_n(a) = f_n(a)/(1-q^a)$ and $h_n(a) = f_n(a)/a$. The following statements are true:*

(a) *f_n is increasing on $[0, c/2]$ and decreasing on $[c/2, c]$.*
(b) *g_n and h_n are strictly decreasing on $(0, c]$.*
(c) *g_n is strictly concave on $(0, c/2]$.*
(d) *f_n is strictly concave on $(0, c)$.*

Proof. (a) The function $f_1(a) = (1-q^a)(1-q^{c-a})$ is increasing on $[0, c/2]$ and is decreasing on $[c/2, c]$. Now suppose that for some $n \geq 2$ the function f_n has the same property. Since $(\alpha;q)_{n+1} = (\alpha;q)_n(1-\alpha q^n)$ we can write

$$f_{n+1}(a) = f_n(a) r_n(a), \quad \text{where } r_n(a) = (1-q^{a+n})(1-q^{c-a+n}). \tag{2.2}$$

Observe that for $n \in \{1, 2, \ldots\}$ the function $a \mapsto r_n(a)$ is increasing on $[0, c/2]$ and is decreasing on $[c/2, c]$. Thus $a \mapsto f_{n+1}(a)$ has the same property, and by mathematical induction the required result follows.

(b) First suppose that $a \in (0, c/2]$. Since

$$\Gamma_q(\alpha+n) = \frac{(q^\alpha;q)_n}{(1-q)^n}\Gamma_q(\alpha),$$

we can rewrite $g_n(a)$ as $g_n(a) = u(a)v_n(a)$, where

$$u(a) = \frac{1}{\Gamma_q(a+1)\Gamma_q(c-a)} \quad \text{and} \quad v_n(a) = (1-q)^{2n-1}\Gamma_q(a+n)\Gamma_q(c-a+n).$$

Now, taking the logarithmic derivative of $u(a)$ and $v_n(a)$, we obtain

$$(\log u(a))' = \psi_q(c-a) - \psi_q(a+1) < 0, \quad a > 0 \geq (c-1)/2,$$
$$(\log v_n(a))' = \psi_q(a+n) - \psi_q(c-a+n) \leq 0,$$

where we have used the fact that the q-digamma function $x \to \psi_q(x) = \Gamma_q'(x)/\Gamma_q(x)$ is increasing on $(0, \infty)$. Thus, g_n is strictly decreasing on $(0, c/2]$ as a product of a decreasing and a strictly decreasing function.

Now, if $a \in [c/2, c]$, from part (a) f_n is decreasing and hence g_n is strictly decreasing as a product of a decreasing and a strictly decreasing function.

Finally, let us consider the functions $w, s : (0, 1] \to \mathbb{R}$, defined by $w(a) = (1-q^a)/a$ and $s(a) = q^a - aq^a(\log q) - 1$. Since $s'(a) = -a(\log q)^2 q^a < 0$ for all $a \in (0, 1]$, the function s is decreasing and hence $s(a) < \lim_{a\to 0} s(a) = 0$ for all $a \in (0, 1]$. Thus $w'(a) = s(a)/a^2 < 0$ for all $a \in (0, 1]$, that is, the function w is strictly decreasing on $(0, 1]$. Clearly, w is also strictly decreasing on $(0, c]$. This in turn implies that the function $a \mapsto h_n(a)$, expressed as $h_n(a) = g_n(a)w(a)$, is strictly decreasing on $(0, c]$ as a product of two strictly decreasing and positive functions.

(c) Since $g_1(a) = f_1(a)/(1 - q^a) = (1 - q^{c-a})$ satisfies $g_1''(a) = -q^{c-a}(\log q)^2 < 0$, we get that g_1 is strictly concave on $(0, c]$. Now, suppose that g_n is also strictly concave for some $n \geq 2$. From (2.2) we have $g_{n+1}(a) = g_n(a)r_n(a)$, and consequently

$$g_{n+1}''(a) = g_n''(a)r_n(a) + 2g_n'(a)r_n'(a) + g_n(a)r_n''(a) < 0$$

because g_n is strictly decreasing from part (b), and r_n is increasing and strictly concave on $(0, c/2]$. Hence the required result follows by using mathematical induction.

(d) Since $f_n(a) = f_n(c - a)$, it is enough to prove the concavity of $a \mapsto f_n(a)$ on $(0, c/2]$. Observe that $a \mapsto 1 - q^a$ is strictly increasing and strictly concave on $(0, 1)$ and so is on $(0, c/2]$. Moreover, recall that in view of parts (b) and (c) g_n is strictly decreasing and strictly concave on $(0, c/2]$. Consequently, for all $n \in \{1, 2, \ldots\}$, $q \in (0, 1)$ and $a \in (0, c/2]$ we have

$$\begin{aligned} f_n''(a) = \big((1 - q^a)g_n(a)\big)'' &= (1 - q^a)''g_n(a) \\ &\quad + 2(1 - q^a)'g_n'(a) + (1 - q^a)g_n''(a) < 0. \end{aligned}$$

With this the proof is complete. □

The first part of the next theorem is the q-version of [2, Theorem 2.3].

Theorem 2.1. *Let $0 < q, x < 1$ and $0 < a < c \leq 1$. The following assertions are valid:*

(a) *$a \mapsto \phi_a(x)$ is strictly concave and strictly sub-additive on $(0, c]$.*
(b) *$a \mapsto \phi_a(x)/(1 - q^a)$ is strictly concave on $(0, c/2]$ and strictly sub-additive on $(0, c]$.*

In particular, for all $a_1, a_2 \in (0, c)$ and $q, x \in (0, 1)$, we have

$$\sqrt{\phi_{a_1}(x)\phi_{a_2}(x)} \leq \frac{\phi_{a_1}(x) + \phi_{a_2}(x)}{2} \leq \phi_{\frac{a_1+a_2}{2}}(x) \leq \phi_{\frac{a_1}{2}}(x) + \phi_{\frac{a_2}{2}}(x). \tag{2.3}$$

Similarly, for all $a_1, a_2 \in (0, c/2]$ and $q, x \in (0, 1)$, we have

$$\sqrt{\frac{\phi_{a_1}(x)\phi_{a_2}(x)}{(1 - q^{a_1})(1 - q^{a_2})}} \leq \frac{\frac{\phi_{a_1}(x)}{1-q^{a_1}} + \frac{\phi_{a_2}(x)}{1-q^{a_2}}}{2} \leq \frac{\phi_{\frac{a_1+a_2}{2}}(x)}{1 - q^{\frac{a_1+a_2}{2}}} \leq \frac{\phi_{\frac{a_1}{2}}(x)}{1 - q^{\frac{a_1}{2}}} + \frac{\phi_{\frac{a_2}{2}}(x)}{1 - q^{\frac{a_2}{2}}}. \tag{2.4}$$

Moreover, the first and third inequalities in (2.4) are valid for all $a_1, a_2 \in (0, c)$.

Proof. In view of part (d) of Lemma 2.1 the function $a \mapsto f_n(a)$ is strictly concave on $(0, c)$. Consequently, for all $a_1, a_2 \in (0, c)$, $a_1 \neq a_2$, $q, x \in (0, 1)$ and for $\lambda \in (0, 1)$ we have

$$\begin{aligned} \phi_{\lambda a_1+(1-\lambda)a_2}(x) &= \sum_{n\geq 0} \frac{f_n(\lambda a_1 + (1 - \lambda)a_2)}{(q^c; q)_n(q; q)_n} x^n \\ &> \sum_{n\geq 0} \frac{\lambda f_n(a_1) + (1 - \lambda) f_n(a_2)}{(q^c; q)_n(q; q)_n} x^n \\ &= \lambda\phi_{a_1}(x) + (1 - \lambda)\phi_{a_2}(x) \end{aligned}$$

i.e. the function $a \mapsto \phi_a(x)$ is strictly concave on $(0, c)$. Moreover, since from part (c) of Lemma 2.1 the function $a \mapsto g_n(a)$ is strictly concave, for all $a_1, a_2 \in (0, c/2]$, $a_1 \neq a_2$,

$q, x \in (0, 1)$ and for $\lambda \in (0, 1)$ we have

$$\begin{aligned}\frac{\phi_{\lambda a_1+(1-\lambda)a_2}(x)}{1-q^{\lambda a_1+(1-\lambda)a_2}} &= \sum_{n\geq 0} \frac{g_n(\lambda a_1+(1-\lambda)a_2)}{(q^c;q)_n(q;q)_n}x^n \\ &> \sum_{n\geq 0} \frac{\lambda g_n(a_1)+(1-\lambda)g_n(a_2)}{(q^c;q)_n(q;q)_n}x^n \\ &= \lambda\frac{\phi_{a_1}(x)}{1-q^{a_1}} + (1-\lambda)\frac{\phi_{a_2}(x)}{1-q^{a_2}}\end{aligned}$$

i.e. the function $a \mapsto \phi_a(x)/(1-q^a)$ is strictly concave on $(0, c/2]$.

Now, from part (b) of Lemma 2.1 the function $a \mapsto h_n(a)$ is strictly decreasing on $(0, c]$, which implies that $a \mapsto f_n(a)$ is strictly sub-additive on $(0, c]$. From this, for $a_1, a_2 \in (0, c]$, $a_1 \neq a_2$ and $q, x \in (0, 1)$ we get

$$\phi_{a_1+a_2}(x) = \sum_{n\geq 0} \frac{f_n(a_1+a_2)}{(q^c;q)_n(q;q)_n}x^n < \sum_{n\geq 0} \frac{f_n(a_1)+f_n(a_2)}{(q^c;q)_n(q;q)_n}x^n = \phi_{a_1}(x)+\phi_{a_2}(x)$$

i.e. the function $a \mapsto \phi_a(x)$ is strictly sub-additive. Similarly, from part (b) of Lemma 2.1 the function $a \mapsto g_n(a)$ is strictly decreasing on $(0, c]$ for $n \in \{1, 2, \ldots\}$, and thus $a \mapsto g_n(a)/a$ is strictly decreasing too on $(0, c]$ as a product of two strictly decreasing functions. This implies that $a \mapsto g_n(a)$ is strictly sub-additive on $(0, c]$. From this, for $a_1, a_2 \in (0, c]$, $a_1 \neq a_2$ and $q, x \in (0, 1)$ we obtain

$$\frac{\phi_{a_1+a_2}(x)}{1-q^{a_1+a_2}} = \sum_{n\geq 0} \frac{g_n(a_1+a_2)}{(q^c;q)_n(q;q)_n}x^n < \sum_{n\geq 0} \frac{g_n(a_1)+g_n(a_2)}{(q^c;q)_n(q;q)_n}x^n = \frac{\phi_{a_1}(x)}{1-q^{a_1}} + \frac{\phi_{a_2}(x)}{1-q^{a_2}}$$

i.e. the function $a \mapsto \phi_a(x)/(1-q^a)$ is strictly sub-additive.

Finally, observe that the first inequalities in (2.3) and (2.4) follow directly from the arithmetic mean–geometric mean inequality, or we can use the known fact that the concavity is stronger than the log-concavity. □

Concluding remark. It is important to mention here that the condition $c \leq 1$ in Lemma 2.1 and Theorem 2.1 is not necessary. Following the proof of Lemma 2.1 it is clear that the function u in the proof of part (b) is also strictly decreasing when $c > 1$ and $a > c_0 = (c-1)/2$. Consequently, following the proof of Lemma 2.1 and Theorem 2.1 it can be shown that for all $n \in \{1, 2, \ldots\}$ we have that g_n and h_n are strictly decreasing on $(c_0, c] \subset (0, c]$, g_n is strictly concave on $(c_0, c/2] \subset (0, c/2]$ and f_n is strictly concave on $(c_0, c_0+1) \subset (0, c)$. Consequently, the function $a \mapsto \phi_a(x)$ is strictly concave on (c_0, c_0+1) and strictly sub-additive on $(c_0, c]$, while the function $a \mapsto \phi_a(x)/(1-q^a)$ is strictly concave on $(c_0, c/2]$ and strictly sub-additive on $(c_0, c]$. These in turn imply that (2.3) is also valid when $c > 1$, $a_1, a_2 \in (c_0, c_0+1)$ and $q, x \in (0, 1)$. Moreover, the first inequality in (2.3) is valid for all $a_1, a_2 \in (0, c)$, and the third inequality in (2.3) holds true for all $a_1, a_2 \in (c_0, c]$. Similarly, the inequality (2.4) is valid for $c > 1$, $a_1, a_2 \in (c_0, c/2]$ and $q, x \in (0, 1)$. Moreover, the first inequality in (2.4) is valid for all $a_1, a_2 \in (0, c)$, and the third inequality in (2.4) holds true for all $a_1, a_2 \in (c_0, c]$.

3. Turán type inequalities for the q-Kummer hypergeometric function

Replacing x by $(1-q)x/(1-b)$ and setting $b=0$ in (1.2), we get the q-Kummer confluent hypergeometric function defined as

$$\phi(q^a,q^c;q,x) := {}_1\phi_1(q^a,q^c;q,(1-q)x) = \sum_{n\geq 0}\frac{(q^a;q)_n(1-q)^n}{(q^c;q)_n(q;q)_n}x^n,$$

which for $q\nearrow 1$ gives the confluent hypergeometric function

$$\phi(a,c;x) := {}_1\phi_1(a,c;x) = \sum_{n\geq 0}\frac{(a)_n}{(c)_n n!}x^n.$$

The following theorem is the q-version of some of the results of [3, Theorem 2].

Theorem 3.1. *Let $q\in(0,1)$. If $a\geq c>0$ and $x>0$, then the function $\omega\mapsto\phi(q^{a+\omega},q^{c+\omega};q,x)$ is log-convex on $[0,\infty)$. Moreover, if $a,c>0$ and $x>0$, then $\omega\mapsto\phi(q^a,q^{c+\omega};q,x)$ is log-convex too on $[0,\infty)$. In particular, the following Turán type inequality is valid for all $a\geq c>0$, $x>0$ and $q\in(0,1)$*

$$\left(\phi(q^{a+1},q^{c+1};q,x)\right)^2\leq\phi(q^a,q^c;q,x)\phi(q^{a+2},q^{c+2};q,x).$$

Moreover, if $a,c>0$, $x>0$ and $q\in(0,1)$, then the next Turán type inequality holds

$$\left(\phi(q^a,q^{c+1};q,x)\right)^2\leq\phi(q^a,q^c;q,x)\phi(q^a,q^{c+2};q,x).$$

Proof. For convenience let us write $\phi(q^a,q^c;q,x)$ as

$$\phi(q^a,q^c;q,x) = \sum_{n\geq 0}r_n(a,c)\frac{(1-q)^n}{(q;q)_n}x^n,$$

where

$$r_n(a,c) := \frac{(q^a;q)_n}{(q^c;q)_n} = \frac{\Gamma_q(a+n)\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(c+n)}.$$

Observe that $\partial^2\log r_n(a+\omega,c+\omega)/\partial\omega^2 = \eta(a)-\eta(c)$, where $\eta(x)=\psi_q'(x+\omega+n)-\psi_q'(x+\omega)$ and $n\in\{0,1,\ldots\}$. On the other hand, since for $q\in(0,1)$ and $x>0$ we have

$$\psi_q'''(x) = (\log q)^4\sum_{m\geq 1}\frac{m^3q^{mx}}{1-q^m}>0,$$

it is clear that the function $x\mapsto\psi_q''(x)$ is increasing on $(0,\infty)$. This in turn implies that for all $q\in(0,1)$, $\omega\geq 0$, $n\in\{0,1,\ldots\}$ and $x>0$ we have $\eta'(x)=\psi_q''(x+\omega+n)-\psi_q''(x+\omega)\geq 0$, that is, the function η is increasing on $(0,\infty)$. Therefore, if $a\geq c$, then $\partial^2\log r_n(a+\omega,c+\omega)/\partial\omega^2\geq 0$. In other words, $\omega\mapsto r_n(a+\omega,c+\omega)$ is log-convex on $[0,\infty)$ for all $n\in\{0,1,\ldots\}$ and hence $\omega\mapsto\phi(q^{a+\omega},q^{c+\omega};q,x)$ is log-convex too, as the infinite sum of log-convex functions.

Similarly, observe that $\partial^2\log r_n(a,c+\omega)/\partial\omega^2=\psi_q'(c+\omega)-\psi_q'(c+\omega+n)$ for $n\in\{0,1,\ldots\}$. On the other hand, since for $q\in(0,1)$ and $x>0$ we have

$$\psi_q''(x) = (\log q)^3\sum_{m\geq 1}\frac{m^2q^{mx}}{1-q^m}<0,$$

the function $x \mapsto \psi_q'(x)$ is decreasing on $(0, \infty)$. This in turn implies that for all $q \in (0, 1)$, $\omega \geq 0, n \in \{0, 1, \ldots\}$ and $a, c, x > 0$ we have $\partial^2 \log r_n(a, c+\omega)/\partial\omega^2 \geq 0$. In other words, $\omega \mapsto r_n(a, c+\omega)$ is log-convex on $[0, \infty)$ for all $n \in \{0, 1, \ldots\}$ and hence $\omega \mapsto \phi(q^a, q^{c+\omega}; q, x)$ is log-convex too, as the infinite sum of log-convex functions. □

The next theorem is the second main result of this section. The idea of the proof of this interesting result is taken from [4].

Theorem 3.2. *If $x > 0$, $c > 0$, $\omega \in \mathbb{N}$ and $a \geq \omega - 1$, then the next Turán inequality is valid*

$$\big(\phi(q^a, q^c; q, x)\big)^2 > \phi(q^{a+\omega}, q^c; q, x)\phi(q^{a-\omega}, q^c; q, x). \tag{3.1}$$

Proof. Let us consider the function $f_\omega : (0, \infty) \to \mathbb{R}$, defined by

$$f_\omega(x) := \big(\phi(q^a, q^c; q, x)\big)^2 - \phi(q^{a+\omega}, q^c; q, x)\phi(q^{a-\omega}, q^c; q, x).$$

In view of $(q^{\alpha+1}; q)_{m+1} = (q^{\alpha+1}; q)_m(1 - q^{\alpha+m+1})$ and $(q^\alpha; q)_{m+1} = (q^{\alpha+1}; q)_m(1 - q^\alpha)$ we obtain the contiguous relation

$$\phi(q^{a+1}, q^c; q, x) - \phi(q^a, q^c; q, x) = \frac{q^a(1-q)x}{1-q^c}\phi(q^{a+1}, q^{c+1}; q, x).$$

By using this relation, we obtain

$$\begin{aligned}
f_{\omega+1}(x) - f_\omega(x) &= \phi(q^{a+\omega}, q^c; q, x)\phi(q^{a-\omega}, q^c; q, x) \\
&\quad - \phi(q^{a+\omega+1}, q^c; q, x)\phi(q^{a-\omega-1}, q^c; q, x) \\
&= \phi(q^{a-\omega}, q^c; q, x)\Big(\phi(q^{a+\omega}, q^c; q, x) - \phi(q^{a+\omega+1}, q^c; q, x)\Big) \\
&\quad + \phi(q^{a+\omega+1}, q^c; q, x)\Big(\phi(q^{a-\omega}, q^c; q; x) - \phi(q^{a-\omega-1}, q^c; q, x)\Big) \\
&= \phi(q^{a-\omega}, q^c; q, x)\left(-\frac{q^{a+\omega}(1-q)x}{1-q^c}\phi(q^{a+\omega+1}, q^{c+1}; q, x)\right) \\
&\quad + \phi(q^{a+\omega+1}, q^c; q, x)\left(\frac{q^{a-\omega-1}(1-q)x}{1-q^c}\phi(q^{a-\omega}, q^{c+1}; q, x)\right) \\
&= \frac{(1-q)x}{1-q^c}g_\omega(x)
\end{aligned}$$

where

$$\begin{aligned}
g_\omega(x) &= q^{a-\omega-1}\Big(\phi(q^{a+\omega+1}, q^c; q, x)\phi(q^{a-\omega}, q^{c+1}; q, x)\Big) \\
&\quad - q^{a+\omega}\Big(\phi(q^{a-\omega}, q^c; q, x)\phi(q^{a+\omega+1}, q^{c+1}; q, x)\Big) \\
&= \sum_{n\geq 0}\sum_{k=0}^{n}\frac{(q^{a+\omega+1}; q)_k(q^{a-\omega}; q)_{n-k}(1-q)^n}{(q; q)_k(q; q)_{n-k}} \\
&\quad \times\left(\frac{q^{a-\omega-1}}{(q^c; q)_k(q^{c+1}; q)_{n-k}} - \frac{q^{a+\omega}}{(q^c; q)_{n-k}(q^{c+1}; q)_k}\right)x^n
\end{aligned}$$

$$= \sum_{n\geq 0}\sum_{k=0}^{n} \frac{(q^{a+\omega+1};q)_k(q^{a-\omega};q)_{n-k}(1-q)^n}{(q;q)_k(q;q)_{n-k}(q^{c+1};q)_{n-k}(q^{c+1};q)_k} \times \frac{q^{a-\omega-1}(1-q^{c+k})-q^{a+\omega}(1-q^{c+n-k})}{1-q^c}x^n.$$

In what follows we would like to show that under the hypotheses of the theorem the expression $f_{\omega+1}(x)-f_\omega(x)$ is positive. For this consider the notations

$$Q_{n,k}(q) := \frac{(q^{a+\omega+1};q)_k(q^{a-\omega};q)_{n-k}(1-q)^n}{(q;q)_k(q;q)_{n-k}(q^{c+1};q)_{n-k}(q^{c+1};q)_k}$$

and

$$\Delta_{n,k}(q) := q^{a-\omega-1}(1-q^{c+k})-q^{a+\omega}(1-q^{c+n-k}).$$

Observe that the last expression can be rewritten as

$$\Delta_{n,k}(q) = q^a(q^{-\omega-1}-q^\omega)(1-q^{c+k})-q^{a+c+k+\omega}(1-q^{n-2k}).$$

When n is even we obtain

$$\sum_{k=0}^{n} Q_{n,k}(q)\Delta_{n,k}(q) = \sum_{k=0}^{\frac{n}{2}-1} Q_{n,k}(q)\Delta_{n,k}(q) + \sum_{k=\frac{n}{2}+1}^{n} Q_{n,k}(q)\Delta_{n,k}(q) + Q_{n,\frac{n}{2}}(q)q^a(q^{-\omega-1}-q^\omega)\left(1-q^{c+\frac{n}{2}}\right).$$

Since $q^{-\omega-1} > q^\omega$, the last term, that is, $Q_{n,\frac{n}{2}}(q)q^a(q^{-\omega-1}-q^\omega)(1-q^{c+\frac{n}{2}})$, is positive, and consequently to prove that

$$\sum_{k=0}^{n} Q_{n,k}(q)\Delta_{n,k}(q) > 0, \tag{3.2}$$

it is enough to obtain the positivity of the remaining term, given by

$$R_{n,k}(q) := \sum_{k=0}^{\frac{n}{2}-1} Q_{n,k}(q)\Delta_{n,k}(q) + \sum_{k=\frac{n}{2}+1}^{n} Q_{n,k}(q)\Delta_{n,k}(q).$$

Note that, when n is odd, we arrive at the same situation, and in this case we need also to find the positivity of the above expression. Indeed, if $n = 2m+1$ and if we look at the $(m+1)$th term in the sum which appears in (3.2), then we get that $Q_{2m+1,m+1}(q)\Delta_{2m+1,m+1}(q) > 0$ since

$$\Delta_{2m+1,m+1}(q) = q^a(q^{-\omega-1}-q^\omega)(1-q^{c+m+1})-q^{a+c+m+1+\omega}(1-q^{-1}) > 0.$$

Now, changing the index of summation in the second sum of $R_{n,k}(q)$, we arrive at

$$\begin{aligned} R_{n,k}(q) &= \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \left(Q_{n,k}(q)\Delta_{n,k}(q) + Q_{n,n-k}(q)\Delta_{n,n-k}(q)\right) \\ &= \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \Big(q^a(q^{-\omega-1}-q^\omega)\left((1-q^{c+k})Q_{n,k}(q)+(1-q^{c+n-k})Q_{n,n-k}(q)\right) \\ &\quad + q^{a+c+k+\omega}(1-q^{n-2k})\left(Q_{n,n-k}(q)-Q_{n,k}(q)\right)\Big), \end{aligned}$$

where $[\cdot]$ denotes the greatest integer function. Moreover, if we look at the sum appearing in (3.2) when $n = 2m + 1$ is odd, then we can see that the sum without the middle term becomes exactly as above. Thus, the inequality (3.2) is valid if the above sum is positive for n natural number. On the other hand,

$$\begin{aligned}
Q_{n,n-k}(q) - Q_{n,k}(q) &= \frac{(q^{a+\omega+1};q)_{n-k}(q^{a-\omega};q)_k(1-q)^n}{(q;q)_{n-k}(q;q)_k(q^{c+1};q)_k(q^{c+1};q)_{n-k}} \\
&\quad - \frac{(q^{a+\omega+1};q)_k(q^{a-\omega};q)_{n-k}(1-q)^n}{(q;q)_k(q;q)_{n-k}(q^{c+1};q)_{n-k}(q^{c+1};q)_k} \\
&= \frac{(q^{a+\omega+1};q)_k(q^{a-\omega};q)_k(1-q)^n}{(q;q)_{n-k}(q;q)_k(q^{c+1};q)_k(q^{c+1};q)_{n-k}} \\
&\quad \times \left(\frac{(q^{a+\omega+1};q)_{n-k}}{(q^{a+\omega+1};q)_k} - \frac{(q^{a-\omega};q)_{n-k}}{(q^{a-\omega};q)_k}\right) \\
&= \frac{(q^{a+\omega+1};q)_k(q^{a-\omega};q)_k(1-q)^{2n-2k}}{(q;q)_{n-k}(q;q)_k(q^{c+1};q)_k(q^{c+1};q)_{n-k}} \\
&\quad \times \left(\frac{\Gamma_q(a+\omega+n-k+1)}{\Gamma_q(a+\omega+k+1)} - \frac{\Gamma_q(a-\omega+n-k)}{\Gamma_q(a-\omega+k)}\right).
\end{aligned}$$

Now, consider the function $W : (0,\infty) \to \mathbb{R}$, defined by $W(\alpha) := \Gamma_q(\alpha+n-k)/\Gamma_q(\alpha+k)$. We obtain that if $n-k > k$, that is, $k \le [(n-1)/2]$, then

$$\begin{aligned}
\frac{W'(\alpha)}{W(\alpha)} &= \psi_q(\alpha+n-k) - \psi_q(\alpha+k) \\
&= (\log q)\sum_{m\ge 1}\frac{q^{m(\alpha+n-k)} - q^{m(\alpha+k)}}{1-q^m} > 0.
\end{aligned}$$

This in turn implies that the difference $Q_{n,n-k}(q) - Q_{n,k}(q)$ is positive, as well as the sum $R_{n,k}(q)$, and hence the inequality (3.2) is valid for all n natural number. This yields

$$f_{\omega+1}(x) - f_\omega(x) = \frac{x(1-q)}{(1-q^c)^2}\sum_{n\ge 0}\sum_{k=0}^{n} Q_{n,k}(q)\Delta_{n,k}(q)x^n > 0$$

for all $a \ge \omega > 0$, $x > 0$ and $c > 0$. Consequently, we get

$$f_{\omega+1}(x) = (f_{\omega+1}(x) - f_\omega(x)) + (f_\omega(x) - f_{\omega-1}(x)) + \cdots + (f_1(x) - f_0(x)) > 0$$

for $a \ge \omega > 0$, $x > 0$ and $c > 0$. Since $f_0(x) = 0$, replacing ω by $\omega - 1$, the required result follows. □

Concluding remarks

1. First observe that similar results to those obtained in Theorem 3.1 can be deduced also for the q-hypergeometric function ${}_2\phi_1(q^a, q^b, q^c; q, \cdot)$. More precisely, by using the proof of Theorem 3.1 mutatis-mutandis we can prove the following results: if $q \in (0,1)$, $a \ge c > 0$, $b > 0$ and $x \in (0,1)$, then the function $\omega \mapsto {}_2\phi_1(q^{a+\omega}, q^b, q^{c+\omega}; q, x)$ is log-convex on $[0,\infty)$. Moreover, if $a, b, c > 0$ and $x, q \in (0,1)$, then $\omega \mapsto {}_2\phi_1(q^a, q^b, q^{c+\omega}; q, x)$ is log-convex too on $[0,\infty)$. In particular, the following Turán

type inequality is valid for all $a \geq c > 0$, $b > 0$ and $x, q \in (0, 1)$

$$\left({}_2\phi_1(q^{a+1}, q^b, q^{c+1}; q, x)\right)^2 \leq {}_2\phi_1(q^a, q^b, q^c; q, x)\,{}_2\phi_1(q^{a+2}, q^b, q^{c+2}; q, x). \tag{3.3}$$

Moreover, if $a, b, c > 0$, and $q, x \in (0, 1)$, then the next Turán type inequality holds

$$\left({}_2\phi_1(q^a, q^b, q^{c+1}; q, x)\right)^2 \leq {}_2\phi_1(q^a, q^b, q^c; q, x)\,{}_2\phi_1(q^a, q^b, q^{c+2}; q, x).$$

It is important to mention here that the inequality (3.3) is in fact the q-version of the first inequality in [1, Theorem 2.17], obtained for the Gaussian hypergeometric function. Moreover, we note that the Turán type inequalities for the Gaussian hypergeometric functions were investigated also in the papers [11,12] and it would be of interest investigating q-versions of the results obtained therein.

2. We also note that if we take $c = a + b$ in Theorem 3.2, then we obtain the Turán type inequality

$$\left(\phi(q^a, q^{a+b}; q, x)\right)^2 > \phi(q^{a+\omega}, q^{a+b}; q, x)\phi(q^{a-\omega}, q^{a+b}; q, x), \tag{3.4}$$

where $x > 0$, $b > 0$, $\omega \in \mathbb{N}$ and $a \geq \omega - 1$. Observe that when $q \nearrow 1$, the inequality (3.4) reduces to [4, Theorem 1]

$$(\phi(a, a + b; x))^2 \geq \phi(a - \omega, a + b; x)\phi(a + \omega, a + b; x),$$

where $x > 0$, $b > 0$, $\omega \in \mathbb{N}$ and $a \geq \omega - 1$. Note that the above inequality was proved to be valid in [4] for all $a, b \geq \omega - 1$, $a, b > 0$ and $x \neq 0$.

3. Observe also that if $q \nearrow 1$ in Theorem 3.2, then we get the following result: if $c > 0$, $x > 0$, $\omega \in \mathbb{N}$ and $a \geq \omega - 1$, then the Turán type inequality

$$(\phi(a, c; x))^2 \geq \phi(a - \omega, c; x)\phi(a + \omega, c; x)$$

is valid. We note that this inequality was proved by Barnard et al. [4, Corollary 2] for $c + 1 > 0$, $c \neq 0$, $x > 0$, $\omega \in \mathbb{N}$ and $a \geq \omega - 1$, and in the proof of Theorem 3.2 we used a similar approach to that in [4, Theorem 1].

4. It is also important to mention the inequality [4, Corollary 3]

$$\frac{\phi(a + \omega, a + b; x) + \phi(a - \omega, a + b; x)}{2} > \phi(a, a + b; x) > \sqrt{\phi(a + \omega, a + b; x)\phi(a - \omega, a + b; x)},$$

which is valid for all $x \neq 0$, $\omega \in \mathbb{N}$ and $a, b \geq \omega$. Observe that the q-version of the right-hand side of the above inequality is the following

$$\phi(q^a, q^{a+b}; q, x) > \sqrt{\phi(q^{a+\omega}, q^{a+b}; q, x)\phi(q^{a-\omega}, q^{a+b}; q, x)},$$

which is valid for all $x > 0$, $b > 0$, $\omega \in \mathbb{N}$ and $a \geq \omega - 1$, according to (3.4). Moreover, it would be of interest to find the q-version of the left-hand side of the above mean inequality, obtained in [4, Corollary 3]. Unfortunately, we were not able to determine the sign of the expression

$$\frac{\phi(q^{a+\omega}, q^{a+b}; q, x) + \phi(q^{a-\omega}, q^{a+b}; q, x)}{2} - \phi(q^a, q^{a+b}; q, x).$$

The problem is that the coefficients of the above expression do not have constant sign.

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